MATHEMATICS

MATHEMATICS 31

MODULE 3

The Derivative

Distance Learning

Alberta EDUCATION

Mathematics 31

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THE DERIVATIVE

Distance Learning

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Mathematics 31
Student Module Booklet
Module 3
The Derivative
Alberta Distance Learning Centre
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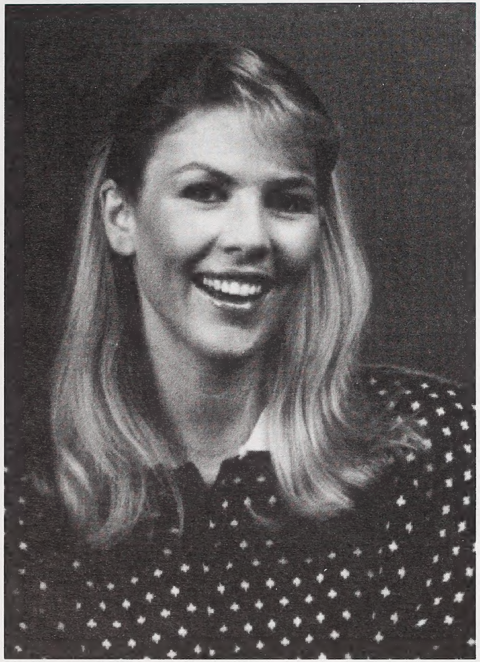
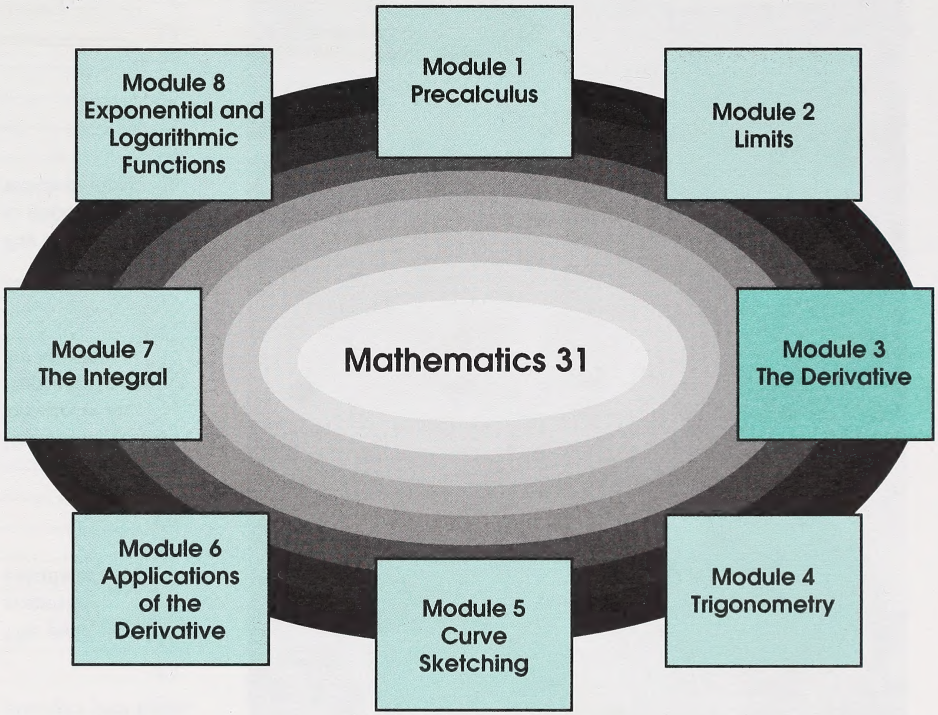
Welcome



WESTFILE INC.

Welcome to Module 3. We hope you'll enjoy your study of The Derivative.

Mathematics 31 contains eight modules. Work through modules in the order given, since several concepts build on each other as you progress in the course.



The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

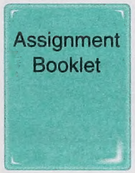
- Use your scientific calculator.

- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.

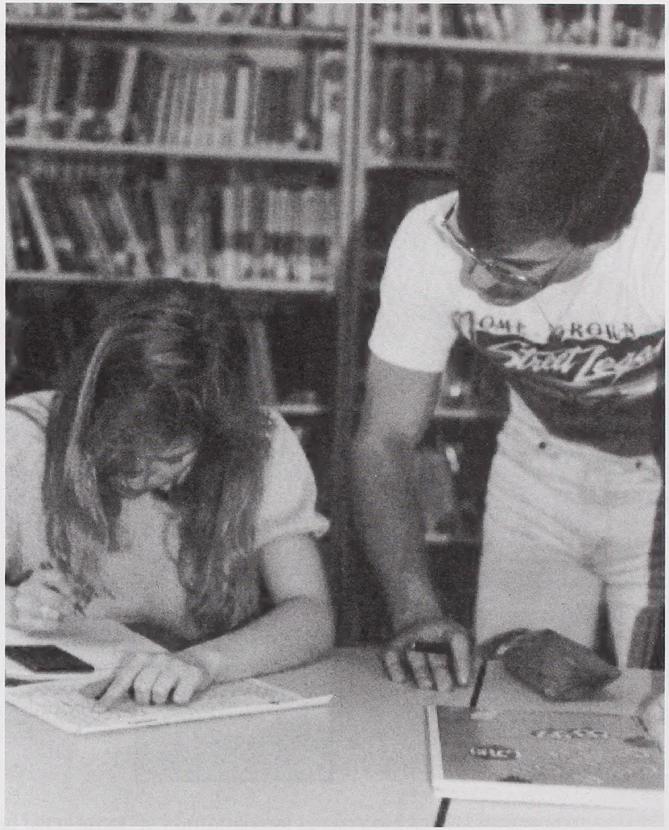
- View a videocassette.

- Pay close attention to important words or ideas.



- Answer the questions in the assignment booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

CONTENTS

Module Overview

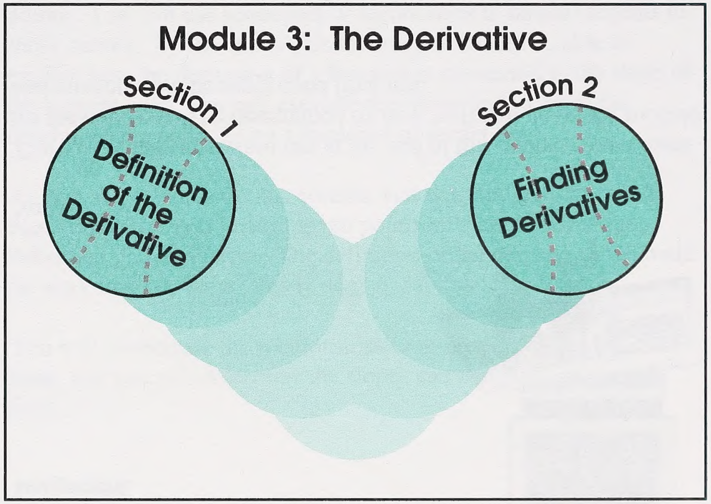
When you see a bouncing ball, you glimpse a world that is in constant change. How fast is the ball moving? What is the ball's position at a given instant of time? What is the maximum height to which it rises? How long will it remain in the air? These are all questions with which calculus is concerned.

But what is calculus? Calculus is the language of the sciences. The word *calculus* comes from the Latin word for "pebble" (pebbles were one of the earliest aids to counting). Two seventeenth century mathematicians, Sir Isaac Newton and Baron von Leibniz, are credited with the invention of calculus. (At one time, there was extensive controversy as to whom the invention of calculus should be attributed. However, both Newton and Leibniz worked independently, were contemporaries, and should be given equal credit.)

There are two branches of calculus: differential calculus and integral calculus. This module introduces you to differential calculus. Differential calculus is concerned with growth and change: the motion of the planets; the growth rates of organisms and their populations; the position, speed, and acceleration of a rocket; the rate at which a cup of coffee cools; the changes in the volume of a container when its dimensions are altered; and so on.

This module consists of two sections. The first section deals with definitions of the derivative – the primary tool of differential calculus. In particular, Section 1 reveals the relationships between the derivative and secants, tangents, and normals drawn to the graphs of functions.

Section 2 deals with the methods of finding derivatives of algebraic functions. These methods are essential to your appreciation of the extent to which differential calculus can be applied in the sciences and in everyday applications.



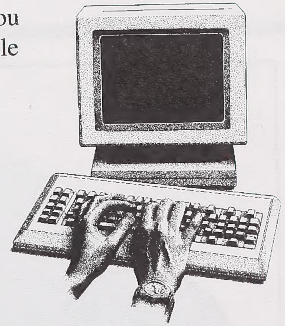
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete two section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**25 marks**
Section 2 Assignment	**55 marks**
Final Module Assignment	**20 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

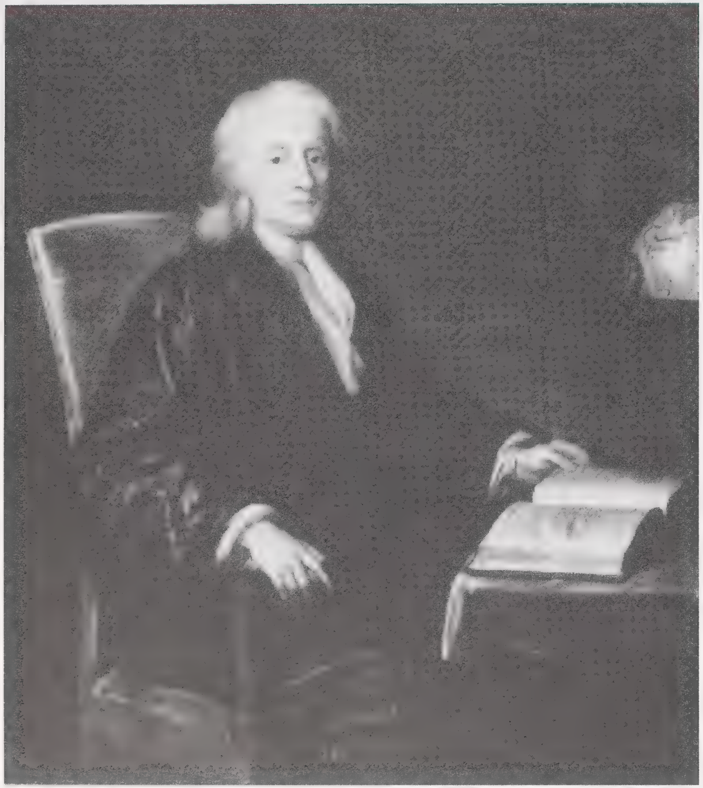
If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Definition of the Derivative



THE BETTMANN ARCHIVE

Sir Isaac Newton is recognized as one of the world's most eminent physicists and mathematicians. His greatest treatise, *Principia*, outlined gravitational theory, laying the basis for classical physics. In 1664, at the age of 22, Newton began his work on calculus, which he called the Theory of Fluxions. Newton used this theory to find tangents and the rate of change of the graphs of functions.

In this section you will find the slopes and equations of secant lines—lines which intersect the graphs of functions in at least two points. You will use sequences of secant lines to obtain tangents to those curves. By the end of this section you should be able to explain how the derivative of a function is connected to the slope of the tangent line. You will estimate the numerical value of the derivative at a point, using a sequence of secant lines.

As part of your study in this section, you will find the slopes and equations of tangent lines at given points on a curve, using the definition of a derivative. The definition of the derivative will build on work that you have already done in Module 2 on limits.

You will investigate the relationships between tangent and normal lines, and you will determine the slopes and equations of normal lines.

Activity 1: Secants

Calculus is the mathematics of motion and change. If you watch an approaching skater, you will notice that not only does the skater's position change from moment to moment, the speed also changes. If the skater takes 6 s to cover a distance of 18 m to pass you, the average speed in that interval is $\frac{18\text{ m}}{6\text{ s}}$ or 3 m/s. But is that the speed the instant the skater passes you? Is that the speed 3 s after the skater passes you? It is very likely that the speed of the skater will be somewhat faster or slower. One approach to the calculation of instantaneous speed involves an analysis of secants of the graph of the function that describes the skater's position. What are secants and what do they have to do with this problem?

Example 1

A speed skater is moving towards you from the left. You wait for the skater to pass. The skater's position s (in metres), relative to you, at time t (in seconds) is described by the function $s=-\frac{1}{3}t^2+5t-12$. Graph the function, and find the skater's average speed between $t=0$ s and $t=6$ s. How is this speed related to the graph?

Solution

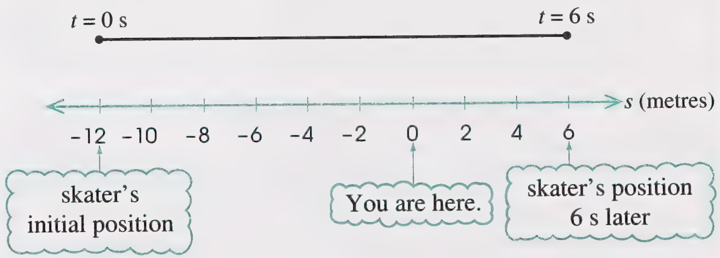
$$\begin{aligned}\text{At } t=0,\ s&=-\frac{1}{3}(0)^2+5(0)-12\\ &=-12\end{aligned}$$

The negative value indicates that the skater is initially 12 m to your left.

$$\begin{aligned}\text{At } t=6,\ s&=-\frac{1}{3}(6)^2+5(6)-12\\ &=6\end{aligned}$$

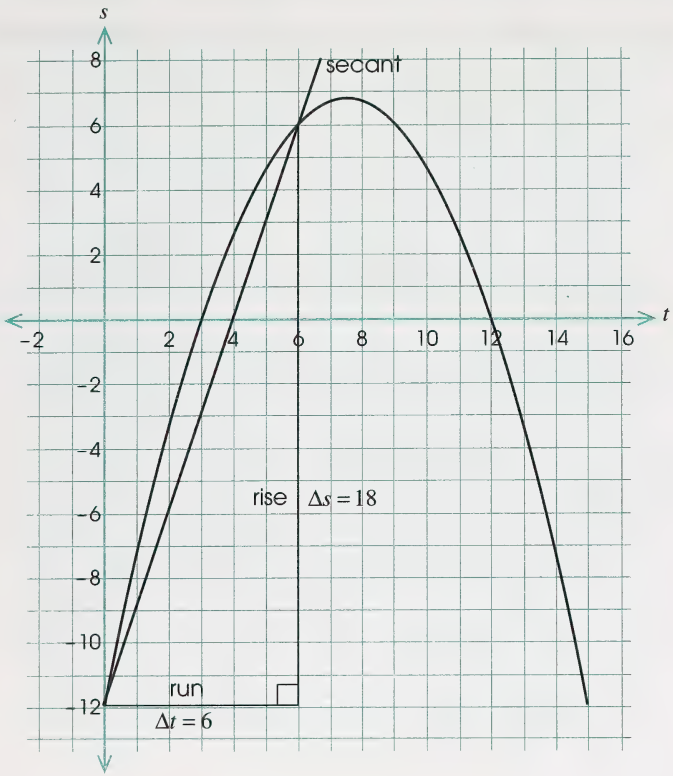
The positive value indicates that after 6 s, the skater is 6 m to your right.

Because the skater is travelling in a straight line, the 18 m covered could be represented on a number line.



$$\begin{aligned}\text{average speed}&=\frac{\text{distance travelled}}{\text{time}}\\ &=\frac{18\text{ m}}{6\text{ s}}\\ &=3\text{ m/s}\end{aligned}$$

However, you can extract more information from the function when it is drawn in the plane. You can readily see the position of the skater at any given time t.



The first ordered pair $(0, -12)$ indicates $s = -12$ when $t = 0$.

The second ordered pair $(6, 6)$ indicates $s = 6$ when $t = 6$.

The line that passes through these points is a secant line. A **secant** is a line that intersects a curve at two or more points. The slope of the secant is determined as follows:

$$\begin{aligned} m_{\text{sec}} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{\Delta s}{\Delta t} \\ &= \frac{18}{6} \\ &= 3 \text{ m/s} \end{aligned}$$

The slope of the secant is the average speed over the time interval, or, in other words, the rate of change of distance with respect to time.

Δs, read **delta *s***, is the skater's change in position.

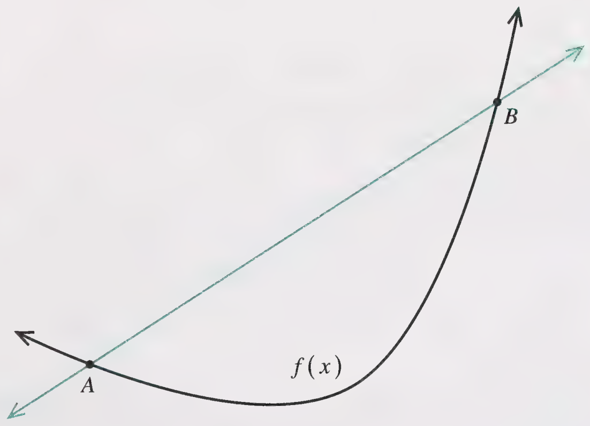
$$\begin{aligned} \Delta s &= \text{final position} - \text{initial position} \\ &= 6 - (-12) \\ &= 6 + 12 \\ &= 18 \text{ m} \end{aligned}$$

Δt, read **delta *t***, is the time interval during which the change in position occurs.

$$\begin{aligned} \Delta t &= \text{final time} - \text{initial time} \\ &= 6 - 0 \\ &= 6 \text{ s} \end{aligned}$$

In the preceding example you saw that a secant is any straight line that intersects a curve at two or more distinct points.

The secant here is AB, and the curve is $f(x)$.

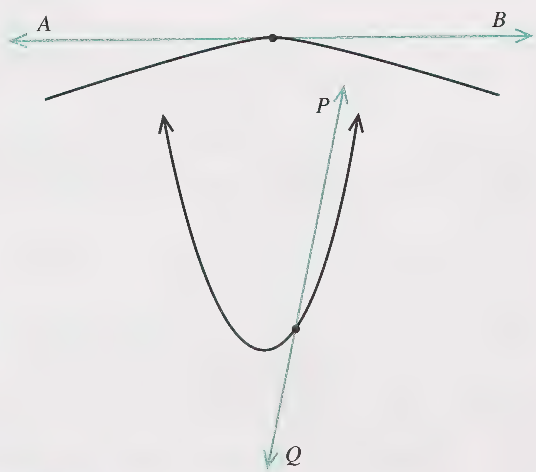


1. a. Draw two curves showing the secant line cutting one curve in two places.

   b. Draw two curves showing the secant line cutting one curve in more than two places.

2. a. Why can't you draw a secant line that intersects a curve at one point?

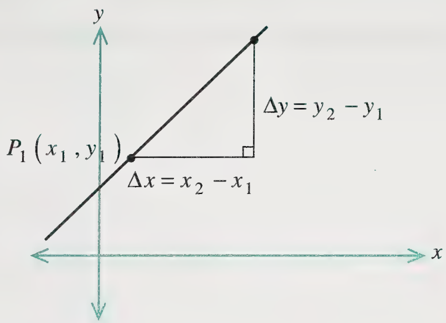
   b. Why are the following lines, AB and PQ, not considered secants?



Check your answers by turning to the Appendix.

In this section, you must be able to find the slope of secant lines. To do this, apply the following formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

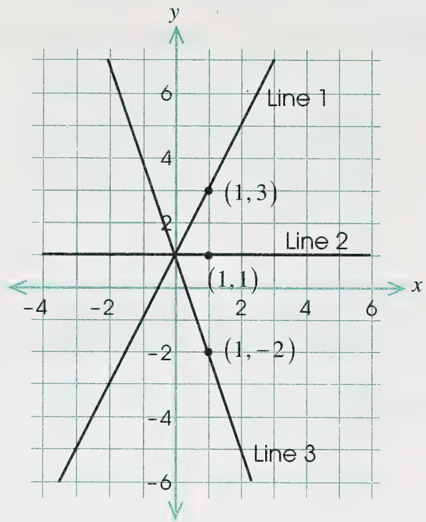


This formula may also be written $m = \frac{\Delta y}{\Delta x}$, where $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$.

Remember: The symbols Δx and Δy are read **delta *x*** and **delta *y***, and represent the change in x and the change in y respectively. The ratio $\frac{\Delta y}{\Delta x}$ (like speed) is the rate of change in y with respect to x.

Example 2

Describe the slope of each line sketched in the given graph.



Solution

Use the formula to find the slope of each line.

Line 1: $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (1, 3)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-1}{1-0} \\ &= \frac{2}{1} \\ &= 2 \end{aligned}$$

In this case, the slope is positive; y increases as x increases. The slope, or the rate of change in y with respect to x, is **y increases by 2 when x increases by 1**. The line rises to the right.

Line 2: $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (1, 1)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-1}{1-0} \\ &= \frac{0}{1} \\ &= 0 \end{aligned}$$

In this case, the slope is zero; y neither increases nor decreases. The line is horizontal. The slope, or the rate of change in y with respect to x, is **y is constant when x increases**.

Line 3: $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (1, -2)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-1}{1-0} \\ &= \frac{-3}{1} \\ &= -3 \end{aligned}$$

In this case, the slope is negative; y decreases as x increases. The slope, or the rate of change in y with respect to x, is **y decreases by 3 when x increases by 1**. The line falls to the right.

3. Find the slope of the line through $(2, 3)$ and $(4, 6)$.

4. Describe the slope of the line $y = -1x + 3$.

Check your answers by turning to the Appendix.

Example 3

Find the slope of the secant of $y = 2x^4$ through each pair of points at which x has the values 1 and 2 respectively.

Solution

Substitute $x = 1$ and $x = 2$ in $y = 2x^4$ to get the corresponding values of y.

$$\begin{aligned} \text{If } x_1 = 1,\ y_1 &= 2(1)^4 \\ &= 2(1) \\ &= 2 \end{aligned}$$

$$\text{If } x_2 = 2,\ y_2 = 2(2)^4$$
$$= 2(16)$$
$$= 32$$

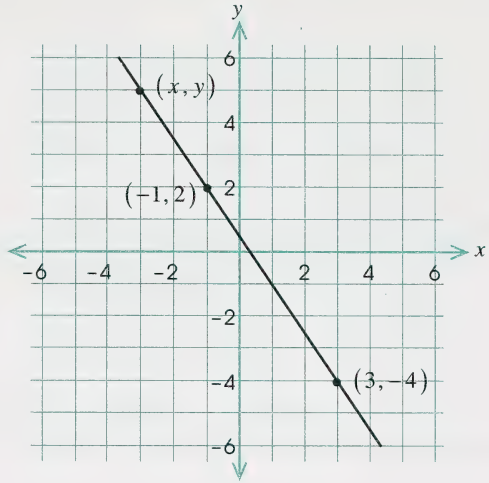
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{32-2}{2-1}$$
$$= \frac{30}{1}$$
$$= 30$$

The slope of the secant is 30.

Example 4

In standard form $(Ax + By + C = 0)$, find the equation of the line passing through $(-1, 2)$ and $(3, -4)$.

Solution



Method 1

Find the slope of the line using the two given points.

$$(x_1, y_1) = (-1, 2) \text{ and } (x_2, y_2) = (3, -4)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-4-2}{3-(-1)}$$

Because the slope of a straight line is constant, any two points on the line could have been used. Find the slope a second time using (x, y) and (x_1, y_1).

$$\begin{aligned} m &= \frac{y-y_1}{x-x_1} \\ &= \frac{y-2}{x-(-1)} \end{aligned}$$

Now, the slopes are equal.

$$\begin{aligned} \frac{y-y_1}{x-x_1} &= \frac{y_2-y_1}{x_2-x_1} \\ \frac{y-2}{x-(-1)} &= \frac{-4-2}{3-(-1)} \\ \frac{y-2}{x+1} &= \frac{-6}{4} \\ \frac{y-2}{x+1} &= \frac{-3}{2} \\ 2(y-2) &= -3(x+1) \\ 2y-4 &= -3x-3 \\ 3x+2y-1 &= 0 \end{aligned}$$

Method 2

Find the slope of the line using the two given points.

$(x_1, y_1) = (-1, 2)$ and $(x_2, y_2) = (3, -4)$

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{-4-2}{3-(-1)} \\ &= \frac{-6}{4} \\ &= -\frac{3}{2} \end{aligned}$$

Substitute the value of the slope into the slope-y-intercept formula for a linear equation.

$$\begin{aligned} y &= mx+b \\ y &= -\frac{3}{2}x+b \end{aligned}$$

The value of b can be found by substituting the coordinates of either point into the equation.

$$\begin{aligned} y &= -\frac{3}{2}x+b \\ 2 &= -\frac{3}{2}(-1)+b \\ 4 &= 3+2b \\ 2b &= 1 \\ b &= \frac{1}{2} \end{aligned}$$

Substitute $(-1, 2)$ into the equation.

$$\therefore\ y = -\frac{3}{2}x+\frac{1}{2}$$

Put this result in standard form.

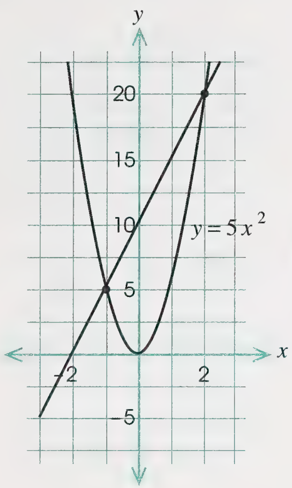
$$2y = -3x + 1$$
$$3x + 2y - 1 = 0$$

Multiply the equation by 2.

Example 5

Determine the equation of the secant of the graph $y = 5x^2$ between the points for which the abscissae are -1 and 2.

Solution



Remember: Abscissae are x-values!

First of all, substitute $x = -1$ and $x = 2$ in $y = 5x^2$ to get the corresponding values of y.

When $x_1 = -1$, $y_1 = 5(-1)^2$
$$= 5(1)$$
$$= 5$$

When $x_2 = 2$, $y_2 = 5(2)^2$
$$= 5(4)$$
$$= 20$$

Method 1

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y - 5}{x - (-1)} = \frac{20 - 5}{2 - (-1)}$$
$$\frac{y - 5}{x + 1} = \frac{15}{3}$$
$$\frac{y - 5}{x + 1} = \frac{5}{1}$$
$$5(x + 1) = 1(y - 5)$$
$$5x + 5 = y - 5$$
$$5x - y + 10 = 0$$

Method 2

First find the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{20-5}{2-(-1)} \\ &= \frac{15}{3} \\ &= 5 \end{aligned}$$

Substitute $m = 5$ into $y = mx + b$.

$$y = 5x + b$$

Evaluate b by substituting the coordinates of one of the two points through which the secant passes into $y = 5x + b$. Substitute the point $(2, 20)$ into the equation to find b.

$$\begin{aligned} 20 &= 5(2) + b \\ 20 &= 10 + b \\ b &= 10 \end{aligned}$$

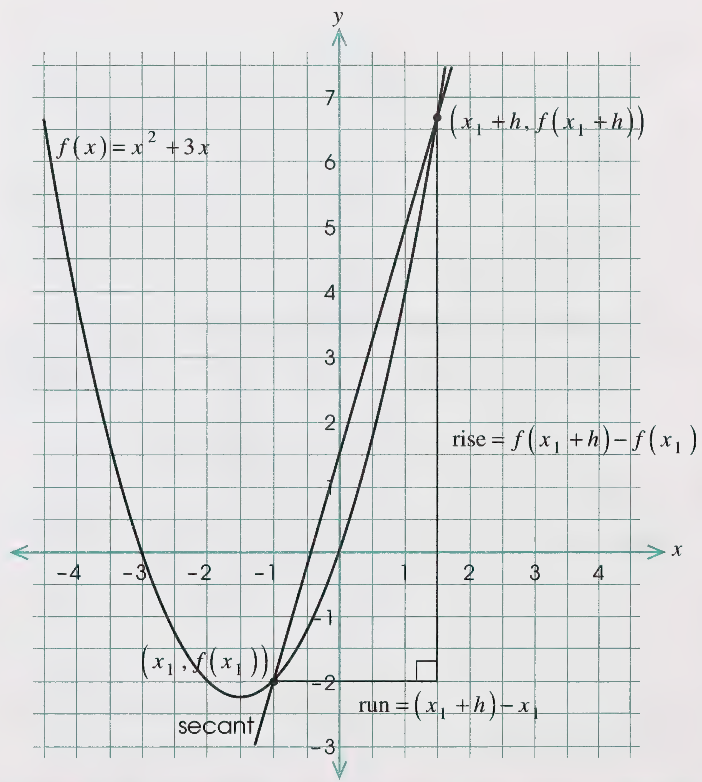
$\therefore\ y = 5x + 10$

In standard form, the equation is $5x - y + 10 = 0$.

Example 6

Determine the slope of the secant intersecting the graph of $f(x) = x^2 + 3x$ at points $(x_1, f(x_1))$ and $(x_1 + h, f(x_1 + h))$.

Solution



$(x_1, y_1) = (x_1, f(x_1))$ and $(x_2, y_2) = (x_1 + h, f(x_1 + h))$

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1} \\
&= \frac{\left[(x_1 + h)^2 + 3(x_1 + h)\right] - \left[x_1^2 + 3x_1\right]}{h} \\
&= \frac{x_1^2 + 2x_1h + h^2 + 3x_1 + 3h - x_1^2 - 3x_1}{h} \\
&= \frac{2x_1h + 3h + h^2}{h} \\
&= \frac{\cancel{h}(2x_1 + 3 + h)}{\cancel{h}} \\
&= 2x_1 + 3 + h
\end{aligned}
$$

5. Points $A(1, -2)$ and $B(2, 7)$ are on the graph of the curve $y = 2x^3 - 3x^2 + 4x - 5$. Find the slope of the secant *AB*.

6. Find the slope of the secant of $y = 5x^3 - 3x^2 - 4x + 7$ through each pair of points at which *x* has the values 1 and -1 respectively.

7. Derive the equations of the secants of the graph $y = x^2 - 3$ between the points which have ordinates at 1 and 6. Determine the equation for each secant on either side of the *y*-axis.

8. **a.** Find the slope of the secants of the curve $y = x^2$ through the point $A(2, 4)$ and the points for which the abscissae are 1.9, 1.99, 1.999, ….

b. If the slopes form a sequence, state the likely slope of the line through *A*.

c. If $A(2, 4)$ is the point (a, b), what is the slope as a multiple of *a*?

9. In Example 6, explain what happens to the slope of the secant line as $h \to 0$.

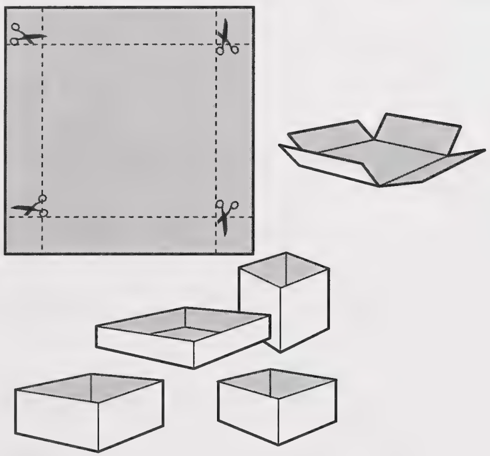
Check your answers by turning to the Appendix.

Calculus is the study of change. Part of this study is the analysis of the slopes of secant lines. A skater's change in position and speed are related to secants, if the skater's position is graphed as a function of time.

Activity 2: Tangents and Normals

If you cut equal squares from each corner of a square sheet of paper and then fold up the sides and join the edges, you make a rectangular box. If you cut a larger square from each corner of that piece of paper, would the volume of the box you make be larger or smaller than the first box? How can you construct a box of maximum volume?

To gain an understanding of this problem, construct a few boxes. Use square sheets of construction paper which have sides of 15 cm.



Investigate the volume of each box. A record of your observations might look like this:

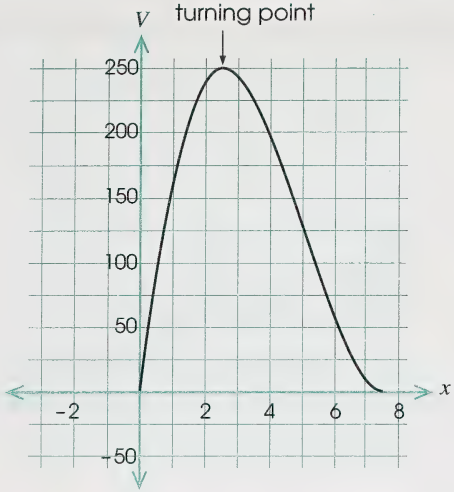
Cutout (cm/side)	Box Length (cm)	Box Width (cm)	Box Height (cm)	Volume $\left(cm^3\right)$
1	13	13	1	169
2	11	11	2	242
3	9	9	3	243
4	7	7	4	196
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
x	$(15-2x)$	$(15-2x)$	x	$(15-2x)(15-2x)(x)$

In this investigation you can see that the volume of each box is given by the polynomial function $V=(15-2x)(15-2x)(x)$, where $0<x\le 7.5$.

The variable x is the independent variable and the volume V is the dependent variable. Simplify the function.

$$\begin{aligned} V &= (15-2x)(15-2x)(x), \text{ where } 0<x\le 7.5 \\ &= \left(225-60x+4x^2\right)x \\ &= 4x^3-60x^2+225x \end{aligned}$$

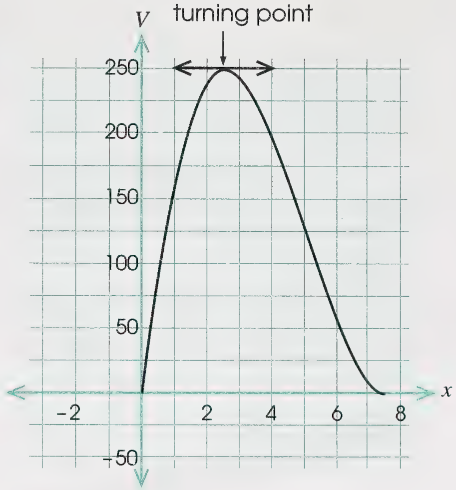
Using techniques from graphing polynomial functions in Mathematics 30 and the values established in your investigation, you can sketch the graph of the function.



It appears from the graph that the maximum volume occurs at the turning point of the graph, when x is approximately 2.5 cm. If you want a more accurate maximum value, you can investigate the interval around 2.5 more closely. Let x be 2.4, 2.5, and 2.6. Calculate the volume (V).

x	V
2.4	249.696
2.5	250.000
2.6	249.704

How can you be sure that 250 is the maximum volume when $x = 2.5$? The method you used to find the maximum is not very accurate, and it is very time-consuming. Take a closer look at the graph.



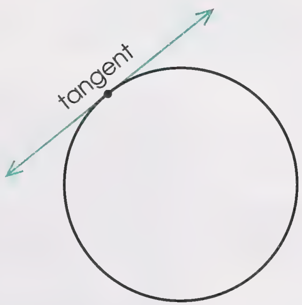
It is clear that the maximum or minimum value of a curve occurs at the turning point of the graph. If you draw a **tangent** or **tangent line** at this turning point, the tangent is always a horizontal line. The slope of a horizontal line is zero.

How can this kind of information help you determine the maximum or minimum value of a curve? The search for the relationship between a function, the slope of a function, and the slope of the tangent to the curve defined by that function led to the development of differential calculus by Newton and Leibniz. You are about to follow their footsteps.

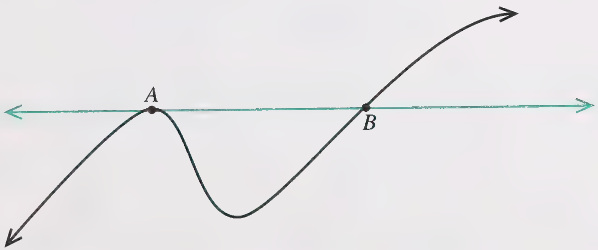
If two points are given, you can determine the slope of a line joining these two points. How do you determine the slope of a tangent to a curve at a given point?

To answer this question, it is necessary to have a closer look at tangents.

Consider a circle. A tangent to the circle is a line that touches the circle at a single point.



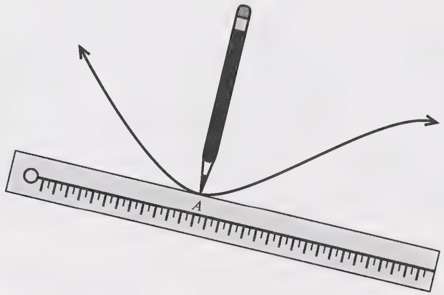
However, when you consider tangents to curves other than circles, a new definition is needed. Look at the following diagram. The tangent touches the curve at A and intersects the curve at B.



What definition can be used? Follow these steps to develop a definition of a tangent.

Step 1: Draw any smooth continuous curve. Choose a point on that curve and label the point A.

Step 2: Now draw a line which is tangent to your curve at point A.

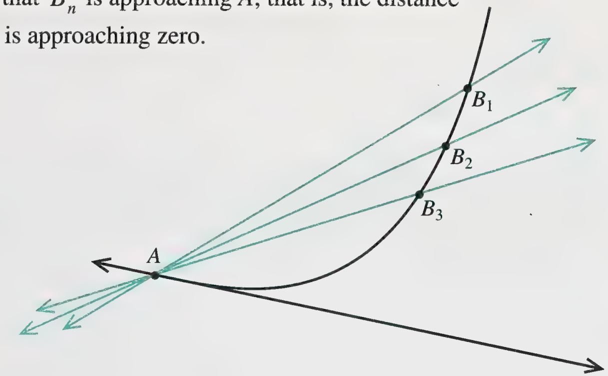


To draw the tangent, you probably placed your ruler on point A and moved it until it **did not** touch the curve at more than one point in the neighbourhood of A.

This procedure indicates a much better definition of a tangent and introduces you to one of the most important concepts of calculus—the **limit**.

If a line passes through two distinct points on a curve, it is called a secant or secant line. In your drawing, the initial positions of the ruler could represent secants. The **tangent** at any Point A on a curve is the **limiting** position of a sequence of secant lines containing A. If you go past that position, the line is once again a secant line.

A tangent to the curve at Point A is the limiting position of a sequence of secants containing Point A and points B_1, B_2, B_3, …, such that B_n is approaching A; that is, the distance AB_n is approaching zero.



Since secants pass through two points on a curve, you can determine the slope of any secant line.

As B_n approaches A, the slope of the secant will approach the value that is the slope of the tangent. Look at the following example.

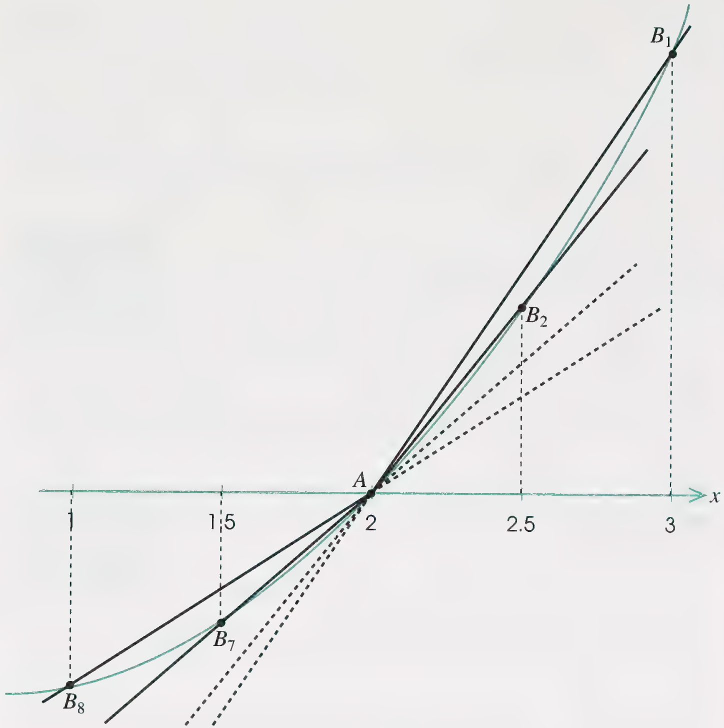
Example 1

If $y=\frac{1}{2}x^2-2$, find the slope of the tangent to the curve at $x=2$.

Solution

When $x=2$, $y=\frac{1}{2}(2)^2-2$
$=0$

Name $(2, 0)$ Point A. Choose a sequence of secants containing A and points B_1, B_2, B_3, …, such that B_n is approaching A.



Choose values of x close to, but not equal to 2. Determine the y-values for the points chosen. Then determine the slopes of the secants through these points. Summarize your findings in a table.

Coordinates of B_n			Slope of secant through $A(2,0)$ and $B_n(x,y)$
Point	x	$y=\frac{1}{2}x^2-2$	
B_1	3	2.5	$\frac{0-2.5}{2-3}=2.5$
B_2	2.5	1.125	$\frac{0-1.125}{2-2.5}=2.25$
B_3	2.1	0.205	$\frac{0-0.205}{2-2.1}=2.05$
B_4	2.01	0.020 05	$\frac{0-0.020\,05}{2-2.01}=2.005$
B_5	1.99	−0.019 95	$\frac{0-(-0.019\,95)}{2-1.99}=1.995$
B_6	1.9	−0.195	$\frac{0-(-0.195)}{2-1.9}=1.95$
B_7	1.5	−0.875	$\frac{0-(-0.875)}{2-1.5}=1.75$
B_8	1	−1.5	$\frac{0-(-1.5)}{2-1}=1.5$

According to the sequence of slopes of the secants, you could conclude that the slope of the tangent at $(2,0)$ is 2. This is the **limiting value** of the sequence of slopes.

At this point you are probably questioning the usefulness of calculus.

Calculus does not seem to provide an easy way of finding the slope of a tangent, nor does it provide a solution to the problem of determining where the slope of the tangent to a curve is zero. You still need to determine how to find maximum and minimum points. More investigations have to be done before you can discover how calculus can make your calculations much easier.

Before continuing with your investigations, do at least one of the following two questions. For each, use the procedure shown in Example 1 to verify the statement.

1. At the point where $x=1$, the slope of the tangent to $y=x^2-3$ is 2.

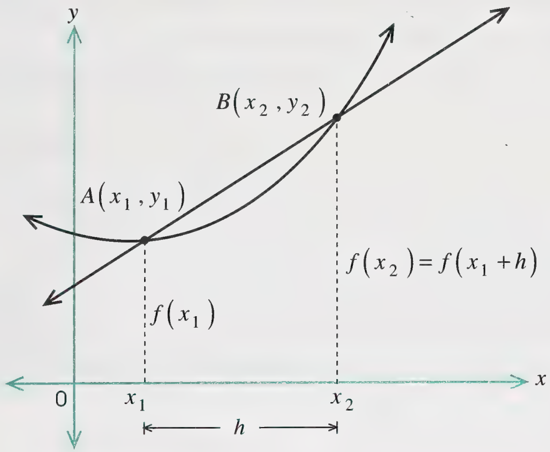
2. At the point where $x=1$, the slope of the tangent to $y=x^3-2x$ is 1.

Check your answers by turning to the Appendix.

How do you determine the slope of the tangent to the curve $y=f(x)$ at $x=x_1$?

If $y_1=f(x_1)$, then the tangent line touches the curve at $A(x_1,y_1)$.

If you move away from Point $A(x_1, y_1)$ a horizontal distance of h units to Point $B(x_2, y_2)$, then $x_2 = x_1 + h$ and $y_2 = f(x_1 + h) = f(x_2)$.



The slope of the secant line is as follows:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1}, \text{ where } h \neq 0$$
$$= \frac{f(x_1 + h) - f(x_1)}{h}$$

Use the symbol Δx to denote the increase in x and Δy to denote the corresponding increase in y.

In this case,

$$\Delta x = x_2 - x_1 = h$$
$$\Delta y = y_2 - y_1 = f(x_1 + h) - f(x_1)$$
$$m_{AB} = \frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}$$

If Point B is moving toward Point A, then h is getting smaller and approaching zero. If $h = 0$, then m_{AB} is undefined; therefore, h cannot be zero. As h approaches zero, the slope of the secant approaches a value that is the slope of the tangent. Zero is the limit of h. The symbol **lim** is used to denote **limit** and the symbol $\boldsymbol{h \to 0}$ is used to denote ***h* approaches zero**.

The slope of the tangent is the limit of the slope of the secant as $h \to 0$.

$$\text{Slope of tangent} = \lim_{h \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

Since x_1 represents different values of x, x will be used instead of x_1.

The value of this limit at x is also called the derivative **of y with respect to x** because it is derived from the original function. The symbol $\frac{dy}{dx}$ is used for this value.

The derivative of y with respect to x can be denoted as follows:

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\Delta y}{\Delta x}$$

This value can also be denoted by other notations:

- $D_x y$
- y' (read as **y prime**)
- $f'(x)$ (read as ***f* prime of *x***)

Since $\frac{dy}{dx}$ represents the first time you take the derivative of y, it should be called the **first derivative** of y. (You can take the derivative of y more than once.) According to the definition of $\frac{dy}{dx}$, you can conclude that **first derivatives are slopes of tangents**.

Example 2

Find the derivative of y with respect to x for the function $f(x)=\frac{1}{2}x^2-2$. Use the derivative to find the slope of the tangent to this curve at $x=2$. Graph the original curve $y=f(x)$ and its tangent at $x=2$.

Solution

$$f(x)=\frac{1}{2}x^2-2 \text{ and } f(x+h)=\frac{1}{2}(x+h)^2-2$$
$$=\frac{1}{2}x^2+\frac{2hx}{2}+\frac{1}{2}h^2-2$$
$$=\frac{1}{2}x^2+hx+\frac{1}{2}h^2-2$$

The derivative is defined as follows:

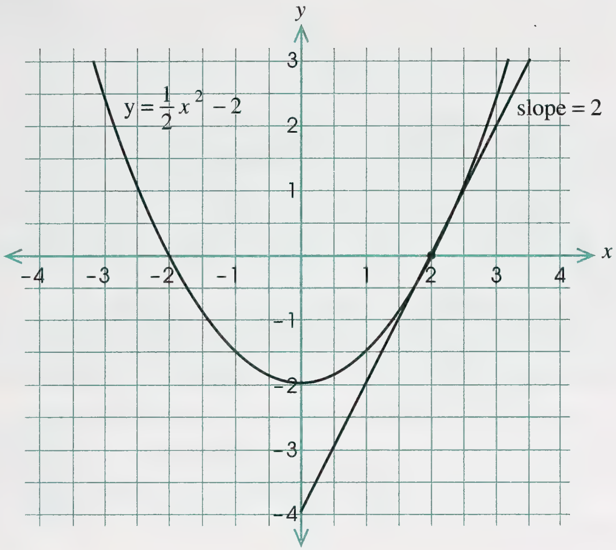
$$\begin{aligned}\frac{dy}{dx}&=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\ &=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad (\text{since } \Delta x=h)\\ &=\lim_{h\to 0}\frac{\frac{1}{2}x^2+hx+\frac{1}{2}h^2-2-\frac{1}{2}x^2+2}{h}\\ &=\lim_{h\to 0}\frac{\left(hx-\frac{1}{2}h^2\right)}{h}\\ &=\lim_{h\to 0}\frac{\cancel{h}\left(x-\frac{1}{2}h\right)}{\cancel{h}}\\ &=x-\tfrac{1}{2}(0)\\ &=x\end{aligned}$$

Therefore, $\frac{dy}{dx} = x$. When $x = 2$, $\frac{dy}{dx} = 2$.

You can also write $f'(x) = x$.

$\therefore f'(2) = 2$.

The slope of the tangent line at $x = 2$ is 2.

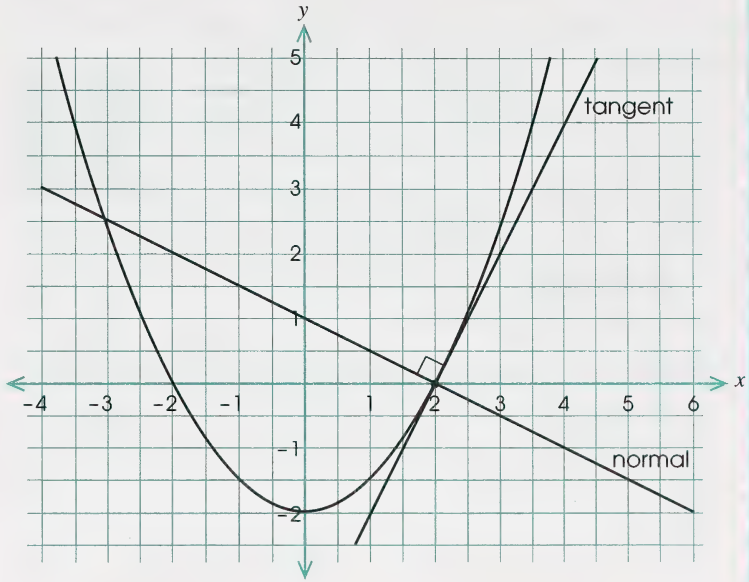


On occasion you will be asked to sketch and determine the equation of a normal to a curve at a given point. A normal to a curve is the line perpendicular to the tangent at its point of contact.

Example 3

Graph the tangent and normal lines at $x = 2$ for the function $f(x) = \frac{1}{2}x^2 - 2$ (given in Example 2). Determine the equation of each.

Solution



The point of contact (tangency) of the tangent to $y = f(x)$ at $x = 2$ is $(2, f(2))$.

$$
\begin{aligned}
f(2) &= \frac{1}{2}(2)^2 - 2 \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The point of contact is $(2, 0)$.

From Example 2, the derivative of y with respect to x is used to find the slope of tangent lines. Because $\frac{dy}{dx} = x$, the slope at $x = 2$ is 2.

Because you now know the slope of the tangent and its point of contact, you may use the formula $y - y_1 = m(x - x_1)$ to find the equation. This formula is called the slope-point formula; as its name suggests, it is used when the slope and a point on the line are known.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 0 &= 2(x - 2) \quad \left(\text{since } m = 2 \text{ and } (x_1, y_1) = (2, 0)\right) \\
y &= 2x - 4 \\
2x - y - 4 &= 0
\end{aligned}
$$

The normal also passes through $(2, 0)$. Because the normal is perpendicular to the tangent, the slope of the normal is the negative reciprocal of the slope of the tangent.

$$
\begin{aligned}
m_{\text{normal}} &= -\frac{1}{m_{\text{tangent}}} \\
&= -\frac{1}{2}
\end{aligned}
$$

As for the tangent, use the slope-point formula.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 0 &= -\frac{1}{2}(x - 2) \quad \left(\text{since } m = -\tfrac{1}{2} \text{ and } (x_1, y_1) = (2, 0)\right) \\
2(y - 0) &= -1(x - 2) \\
2y &= -x + 2 \\
x + 2y - 2 &= 0
\end{aligned}
$$

3. Find $f'(x)$ given $f(x) = x^2 - x$.

4. Find the equations of the tangent and the normal at $x = 1$ to the curve in question 3.

5. The derivative of $f(x) = 2 - x^3$ is $f'(x) = -3x^2$. Find the equations of the tangent and normal at $x = -1$.

Check your answers by turning to the Appendix.

The problem of finding tangents to curves is fundamental to determining maximum and minimum values of functions—functions that model real-world situations such as the volume of a container.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

One of the skills basic to this section is finding a linear equation. You may use various formulas depending on the information given.

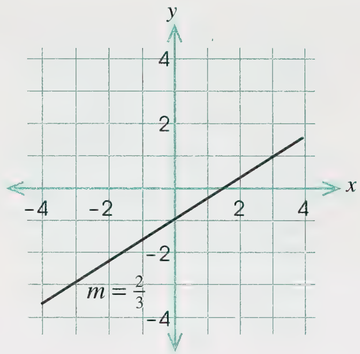
Formula	Information Given
$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$	two points on the line
$y-y_1=m(x-x_1)$	one point on the line and the slope
$y=mx+b$	slope and the *y*-intercept

The slope-*y*-intercept formula, $y=mx+b$, can be easily used in almost all situations.

Example 1

Find the equation of the line with a slope of $\frac{2}{3}$ and a *y*-intercept of -1. Express your answer in the form $Ax+Bx+C=0$.

Solution



The following is the most straight-forward application of the slope-*y*-intercept formula.

Substitute the following values into the formula $y=mx+b$.

$$\text{slope} = m = \frac{2}{3}$$

$$y\text{-intercept} = b = -1$$

$$y=\frac{2}{3}x-1$$

$$3y=3\left(\frac{2x}{3}\right)-3(1)$$

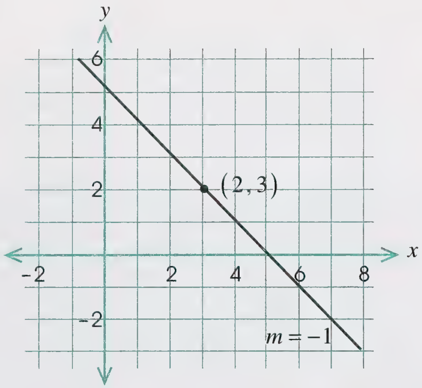
$$3y=2x-3$$

$$2x-3y-3=0$$

Example 2

Determine the equation of the line through $(2, 3)$ and with a slope of -1. Express your answer in the form $Ax + Bx + C = 0$.

Solution



Once again, you can use the formula $y = mx + b$.

$$y = mx + b$$
$$y = -1x + b$$

The slope m is -1.

To calculate the y-intercept b, substitute the coordinates of the given point $(2, 3)$ into the equation.

$$3 = -1(2) + b$$
$$b = 5$$

Now substitute $b = 5$ into the slope-y-intercept equation.

$$y = -1x + 5$$

The required equation is $x + y - 5 = 0$.

Example 3

Find the equation of the line through $(-1, -2)$ and $(3, 2)$. Express your answer in the form $Ax + By + C = 0$.

Solution

First find the slope of the line.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_1 - x_2} \\ &= \frac{2-(-2)}{3-(-1)} \quad \left(\text{where } (x_1, y_1) = (-1, -2) \right. \\ &= \frac{2+2}{3+1} \quad \left. \text{and } (x_2, y_2) = (3, 2)\right) \\ &= \frac{4}{4} \\ &= 1 \end{aligned}$$

Now substitute into $y = mx + b$.

$$y = 1x + b$$

Use either point and replace x and y by its coordinates.

$$2 = 1(3) + b \quad \text{(using } (3, 2)\text{)}$$
$$b = -1$$

Now replace b by -1.

$$y = x - 1$$

In standard form, the equation is $x - y - 1 = 0$.

1. Find the equation of the line with a slope of $-\frac{2}{3}$ and an x-intercept of 1. Express your answer in the form $Ax + By + C = 0$.

2. Determine the equation of the line with a slope of 5 and which passes through $(4, -3)$. Express your answer in the form $Ax + By + C = 0$.

3. Find the equation of the line through $(1, 2)$ and $(3, -2)$. Express your answer in the form $Ax + By + C = 0$.

Check your answers by turning to the Appendix.

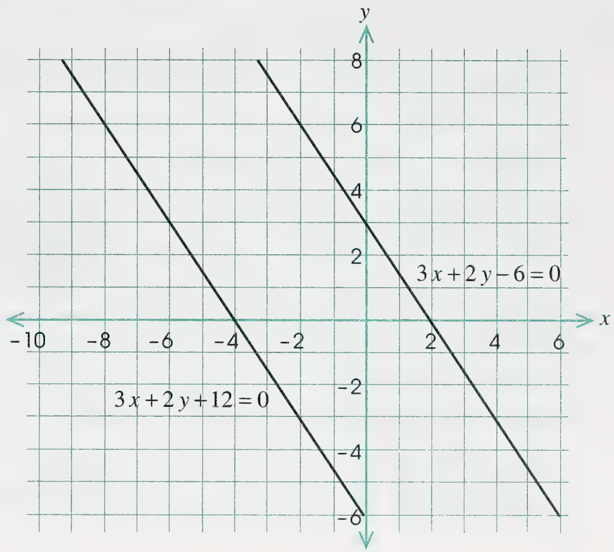
Enrichment

In Activity 2, you found equations of tangents and normals (perpendicular lines). Is there a fast way of finding equations of perpendicular lines or parallel lines?

Example 1

Compare the graphs of $3x + 2y - 6 = 0$ and $3x + 2y + 12 = 0$.

Solution



Express each equation in slope-y-intercept form, $y = mx + b$.

Graph of $3x + 2y - 6 = 0$

$$2y = -3x + 6$$
$$y = -\frac{3}{2}x + 3$$

The slope $= m = -\frac{3}{2}$; and the y-intercept $= b = 3$.

Graph of $3x + 2y + 12 = 0$

$$2y = -3x - 12$$
$$y = -\frac{3}{2}x - 6$$

The slope $= m = -\frac{3}{2}$; and the y-intercept $= b = -6$.

Because the slopes are equal, the lines are parallel. Notice that the terms in x and y are the same in both equations.

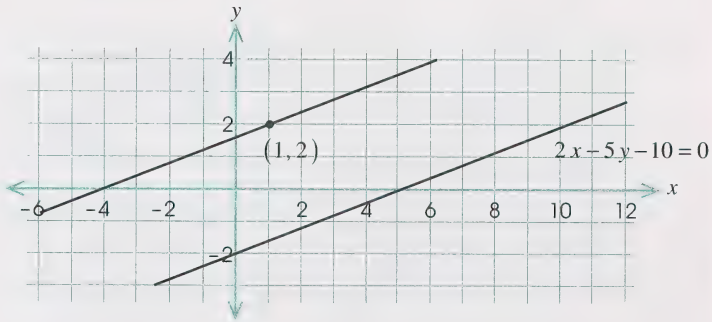
$\boxed{3x + 2y} - 6 = 0$ and $\boxed{3x + 2y} + 12 = 0$

For parallel lines, the terms in x and y will be the same or be multiples. For instance, the graphs of $2x - 3y - 6 = 0$ and $4x - 6y - 7 = 0$ would be parallel. Check by finding their slopes.

Example 2

Find the equation, in standard form, of the line which lies on $(1, 2)$ and is parallel to $2x - 5y - 10 = 0$.

Solution



Because the lines are parallel, the required equation must be of the form $2x - 5y + C = 0$. The only difference between the equations is in the constant terms. To evaluate C, substitute the coordinates of the given point into this form.

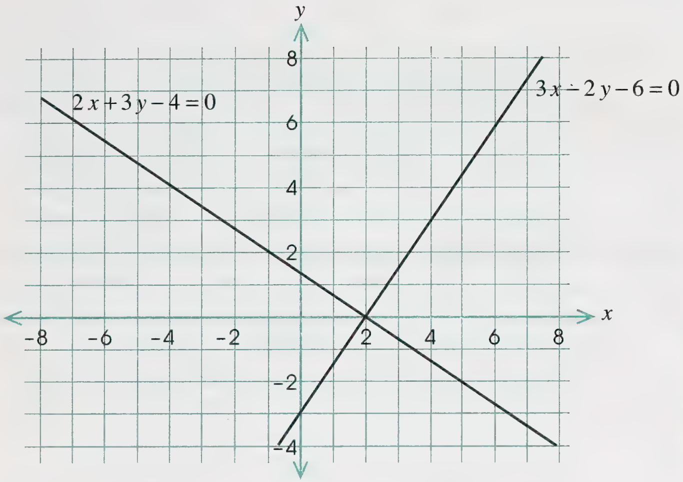
$$2(1) - 5(2) + C = 0 \quad (\text{Replace } x \text{ by 1 and } y \text{ by 2.})$$
$$2 - 10 + C = 0$$
$$C = 8$$

Therefore, the required equation is $2x - 5y + 8 = 0$.

Example 3

Graph $3x-2y-6=0$ and $2x+3y-4=0$. Find the slope of each line. How are they related?

Solution



Express each equation in slope-y-intercept form, $y=mx+b$.

$$\begin{aligned} 2x+3y-4&=0\\ 3y&=-2x+4\\ y&=-\frac{2}{3}x+\frac{4}{3} \end{aligned}$$

The slope of $2x+3y-4=0$ is $-\frac{2}{3}$.

$$\begin{aligned} 3x-2y-6&=0\\ 3x-6&=2y\\ y&=\frac{3}{2}x-3 \end{aligned}$$

The slope of $3x-2y-6=0$ is $\frac{3}{2}$.

The slopes are negative reciprocals when the lines are perpendicular. Notice that, in the equations of perpendicular lines, the coefficients of the x and y terms have been interchanged, and the sign between those terms has been changed:

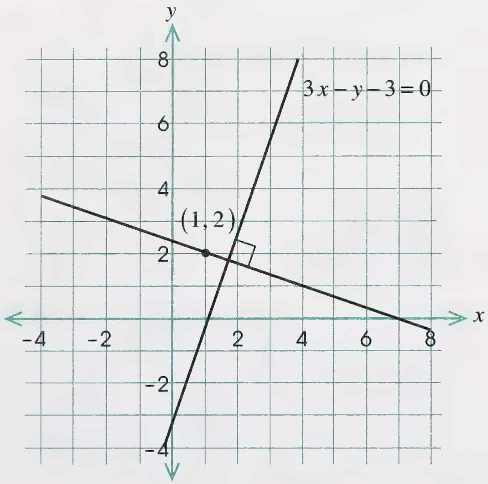
$$\mathbf{2}x+\mathbf{3}y-4=0 \text{ and } \mathbf{3}x-\mathbf{2}y-6=0$$

In general, $Ax+By+C_1=0$ is perpendicular to $Bx-Ay+C_2=0$.

Example 4

Find the equation of the line on $(1,2)$ and perpendicular to $3x-y-3=0$.

Solution



Because the lines are perpendicular, the required equation must be of the form $1x+3y+C=0$.

$$3x-1y-3=0$$

Interchange coefficients of the x and y terms; change the signs between the terms.

$$1x+3y+C=0$$

To evaluate C, substitute the coordinates of the given point into the general form. Replace x by 1 and y by 2.

$$\begin{aligned} 1(1)+3(2)+C&=0 \\ 1+6+C&=0 \\ C&=-7 \end{aligned}$$

Therefore, the required equation is $x+3y-7=0$.

1. Determine the equation of the line through $(1, 5)$ and parallel to $x+2y+1=0$.

2. Determine the equation of the line on $(-2, 3)$ and parallel to $2x-y-1=0$.

3. Determine the equation of the line through $(1, 5)$ and perpendicular to $x+2y+1=0$.

4. Determine the equation of the line on $(-2, 3)$ and perpendicular to $2x-y-1=0$.

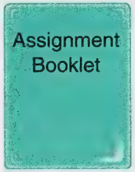
Check your answers by turning to the Appendix.

Conclusion

In this section, you explored the relationships among secants, tangents and normals, and derivatives. In particular, you found the slopes and equations of secant lines; you showed that the slope of a tangent line can be viewed as the limit of the slopes of a sequence of secant lines; and you defined the derivative of a function as the slope of a tangent at a point on the graph of the function. You also estimated the numerical value of the derivative at a point using a sequence of secant lines, and found the slopes and equations of tangent lines at given points on a curve, using the definition of a derivative. Finally, you used the relationship between the slopes of the tangents and normals to find, from the derivative, the slopes and equations of normal lines.

This section retraces the steps Sir Isaac Newton took in determining the slopes of tangents and the curvature of graphs. In 1665, at the age of twenty-two, Newton began his work in calculus; and, in 1666, turned his attention to gravitation and the further development of calculus as a mathematical tool to investigate gravity. The publication of the work he pioneered in calculus, entitled *The Method of Fluxions and Infinite Series*, did not appear until nine years after his death, although many other mathematicians were familiar with the progress he had made well before its publication.

Assignment



You are now ready to complete the section assignment.

Section 2: Finding Derivatives

THE BETTMANN ARCHIVE

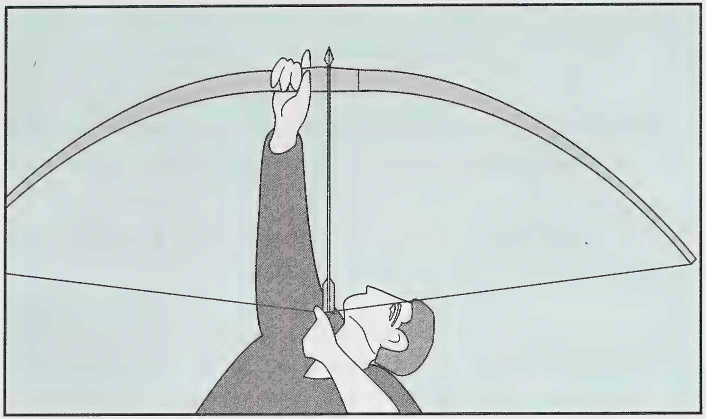
The other inventor of calculus, Gottfried Willhelm, Baron von Leibniz[1], born in Leipzig, Germany, in 1646, began his work in 1673, and developed the subject independently from Newton, using a very different approach to the discipline. Nevertheless, he arrived at essentially the same results. Not only did Leibniz contribute to calculus theory, the symbolism he introduced is still used today. You will be using Leibniz's notation throughout this section.

In Section 2, you will study the basic methods of differentiating algebraic functions. These functions will be expressed both explicitly and implicitly. You will begin by finding the derivative from first principles; that is, you will work from the basic definition. You will then apply a number of theorems that will simplify your work. These rules include the power rule, the rule for the product of a power and a constant, the sum and difference rules, the product rule, the quotient rule, and the chain rule.

Throughout this section, you will be interpreting your results graphically, and you will extend your work with tangent and normal lines. Your study of the first derivative and higher-order derivatives will lay a basis for applying differential calculus to everyday and scientific applications.

[1] His name is also spelt Leibnitz; however, the preferred modern German spelling is Leibniz.

Activity 1: First Principles



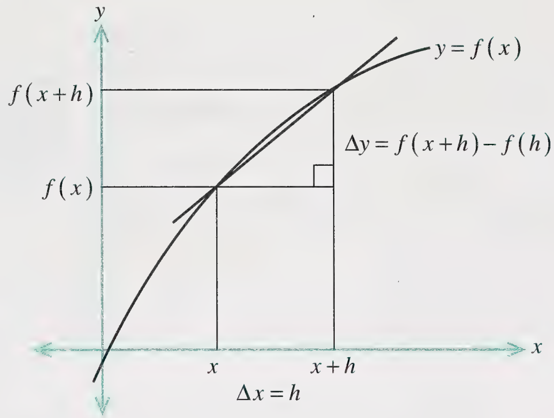
An arrow is shot straight upward at a speed of 40 m/s. If its height h (in metres) after time t (in seconds) is described by $h = -5t^2 + 40t + 2$, how can you use the derivative of this function to find the arrow's speed at a particular instant of time? What does the derivative have to do with a rate such as speed?

Recall from Section 1 that the derivative $\frac{dy}{dx}$ for a function $y = f(x)$ was defined as follows:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (\text{where } \Delta x = h)$$

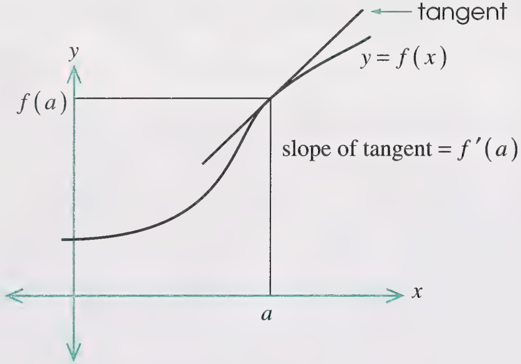
What does the derivative represent in terms of the graph of $y = f(x)$?



Recall that the expression $\frac{f(x+h)-f(x)}{h}$ represents the slope of a secant.

As $h \to 0$, the slope of the secant approaches the slope of the tangent at $(x, f(x))$.

To find the slope of a particular tangent line at, for example, $x = a$, the derivative is evaluated at a.



If $\frac{dy}{dx} = f'(x)$, then the slope of the tangent drawn to $y = f(x)$ at $x = a$ is $f'(a)$.

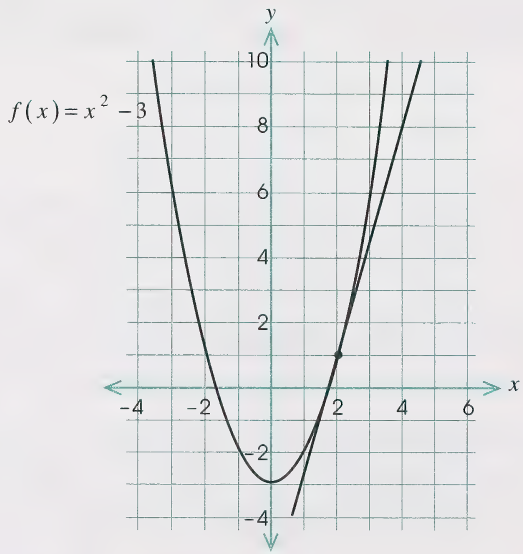
The process of finding the derivative of a function is called **differentiation**. Differentiating a function, or using the definition of the derivative, is called finding the derivative from **first principles**.

Example 1

Find the derivative of $f(x) = x^2 - 3$ from first principles.

Determine the slope of the tangent to the curve at $x = 2$. Graph the original function and its tangent.

Solution



First, find $f(x)$ and $f(x+h)$; then use the definition of the derivative.

$$\begin{aligned} f(x) &= x^2 - 3 \\ f(x+h) &= (x+h)^2 - 3 \\ &= x^2 + 2hx + h^2 - 3 \end{aligned}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (\text{where } \Delta x = h)$$

$$= \lim_{h \to 0} \frac{\left(x^2 + 2hx + h^2 - 3\right) - \left(x^2 - 3\right)}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x+h)$$

You know that $h \neq 0$; however, when h approaches 0, $2x+h$ tends to $2x$. Thus, the slope of the tangent to the curve is given by $2x$.

$$\text{slope} = \frac{dy}{dx} = 2x$$

At the point on the curve where $x = 2$, the slope is as follows:

$$\text{slope} = \frac{dy}{dx} = 2(2)$$
$$= 4$$

Where $x = 2$, the slope of the tangent to the curve $y = x^2 - 3$ is 4.

Notice that the derivative of the function $f(x) = x^2 - 3$ is itself a function, namely $f'(x) = 2x$.

In general, the derivative of a function is another function. This new function may be represented by various symbols such as $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}f(x)$, $D_x y$, or $D_x f(x)$. The symbol $\frac{dy}{dx}$ is called Leibniz notation—so named because Gottfried Wilheim von Leibniz, the co-inventor of calculus, introduced it.

In this module you will be using $\frac{dy}{dx}$ and $f'(x)$ primarily. However, by the end of the course you should become familiar with all symbols.

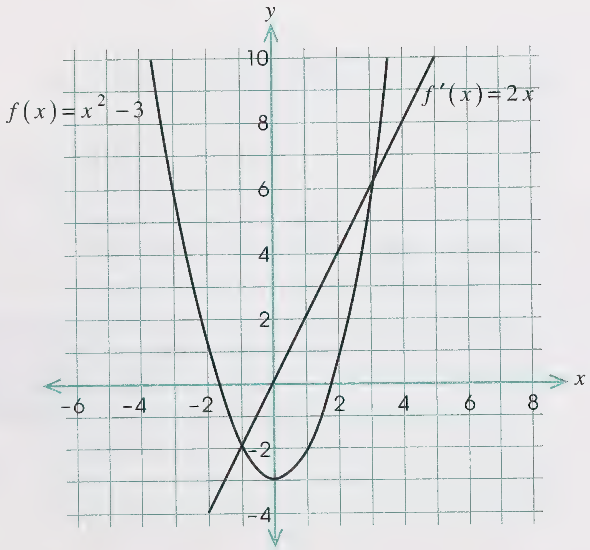
If the derived function f' is defined at $x = a$ (that is, if $f'(a)$ exists), the original function $y = f(x)$ is said to be **differentiable** at a. A function may be differentiable over an interval, or, in the case of $y = x^2 - 3$ from Example 1, may be differentiable over the entire set of real numbers. In Example 1, $f'(x) = 2x$ is defined for all real numbers, or, saying it another way, its domain is the set of real numbers. Of course, the domain of the derived function f' must be a subset of the domain of f itself. After all, f' is obtained from f.

The next two examples should clarify the preceding statements.

Example 2

Graph $f(x)=x^2-3$ and its derived function $f'(x)=2x$. Explain how $f'(x)$ may be interpreted in relation to $f(x)=x^2-3$.

Solution



If you moved a ruler along the curve $f(x)=x^2-3$ from left to right, keeping the ruler tangent to the curve, the slope of the ruler would change. When $x<0$, the slope of the tangent is negative. When $x=0$, the slope of the tangent is zero (the tangent line is horizontal). When $x>0$, the slope is positive. All of this is predicted by $f'(x)=2x$. Consider the following table. Remember; $f'(x)$ is the slope of a tangent.

x	-4	-3	-2	-1	0	1	2	3	4
$f'(x)$	-8	-6	-4	-2	0	2	4	6	8

Example 3

Find the derivative of $y=2x^3$ from first principles.

Solution

$$f(x)=2x^3$$

$$\begin{aligned} f(x+h) &= 2(x+h)^3 \\ &= 2\left(x^3+3hx^2+3h^2x+h^3\right) \\ &= 2x^3+6hx^2+6h^2x+2h^3 \end{aligned}$$

$$\text{slope} = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\cancel{2x^3} + 6hx^2 + 6h^2x + 2h^3\right) - \cancel{2x^3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(6x^2 + 6hx + 2h^2\right)}{\cancel{h}}$$

$$= \lim_{h \to 0} \left(6x^2 + 6hx + 2h^2\right)$$

$$= 6x^2 \qquad \left(\text{When } h \to 0,\ 6hx \to 0 \text{ and } 2h^2 \to 0.\right)$$

1. Find the derivative from first principles for each function.

a. $f(x) = c$ (c is a constant.)
b. $f(x) = x$
c. $f(x) = x^2$
d. $f(x) = x^3$
e. $f(x) = 2x^3 - x + 1$

2. For what values of x are the functions in question 1 differentiable? Explain.

Check your answers by turning to the Appendix.

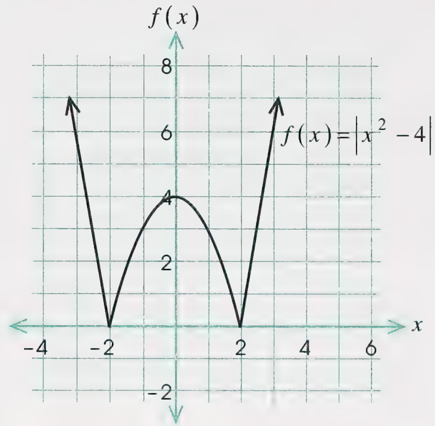
The following example illustrates a function which is **not** differentiable over the entire set of real numbers.

Example 4

Graph $f(x) = \left|x^2 - 4\right|$ using a graphing calculator.

Explain how the graph would change if the absolute value symbol were removed. If possible, find the derivative at $x = 2$ and $x = -2$.

Solution

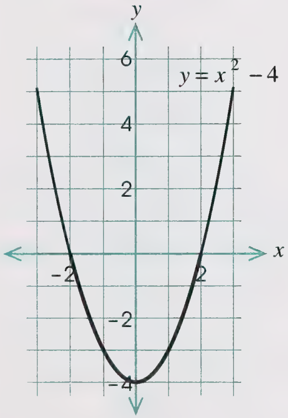


The absolute value symbol keeps the graph on or above the x-axis. $f(x)=\left|x^2-4\right|$ must be non-negative. If the absolute value symbol were absent, the graph would look like Graph A. The absolute value symbol reflects that portion lying below the x-axis (Graph A) into the corresponding portion you see lying above the x-axis (Graph B). Verify this using a table of values.

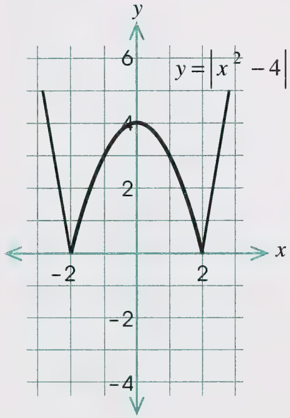
x	−2	−1	0	1	2
y	0	−3	−4	−3	0

x	−2	−1	0	1	2
y	0	3	4	3	0

Graph A



Graph B



What is the derivative at $x=2$? Does $f'(2)$ exist?

Because $\frac{dy}{dx}$ at $x=2$ is a limit $\left(\frac{dy}{dx}=\lim\limits_{h\to 0}\frac{f(2+h)-f(2)}{h}\right)$, it only exists if the left- and right-hand limits are equal.

Calculate the right-hand limit at $x=2$.

$$\begin{aligned} f(2)&=\left|2^2-4\right| \\ &=0 \end{aligned}$$

$$\begin{aligned} f(2+h)&=\left|(2+h)^2-4\right| \\ &=\left|4+4h+h^2-4\right| \\ &=\left|4h+h^2\right| \\ &=4h+h^2 \end{aligned}$$

($h>0$; that is, h approaches 0 from the right.)

$$\begin{aligned} \lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}&=\lim_{h\to 0^+}\frac{4h+h^2-0}{h} \\ &=\lim_{h\to 0^+}(4+h) \\ &=4 \end{aligned}$$

Look at the graph; does this limit seem reasonable?

Calculate the left-hand limit at $x=2$.

$$\begin{aligned} f(2)&=\left|2^2-4\right| \\ &=0 \end{aligned}$$

$$\begin{aligned} f(2+h) &= \left|(2+h)^2 - 4\right| \\ &= \left|4+4h+h^2-4\right| \\ &= \left|4h+h^2\right| \\ &= \left|h(4+h)\right| \\ &= |h| \bullet |4+h| \\ &= -h|4+h| \quad (h<0\text{; that is, } h \text{ approaches 0 from the left.}) \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^-} \frac{-h|4+h|-0}{h} \\ &= \lim_{h\to 0^-} -1|4+h| \\ &= -4 \end{aligned}$$

Look at the graph again. Does this limit seem reasonable? Because that portion of the graph of $y = x^2 - 4$ in the interval $-2 < x < 2$ has been reflected in the *x*-axis by taking the absolute value, the left-hand limit = –(right-hand limit).

Nevertheless, because the left-hand limit is not equal to the right-hand limit, the derivative at $x = 2$ does not exist; that is, $f'(2)$ is not defined. Therefore, $y = f(x)$ is not differentiable at $x = 2$. This function is differentiable at all other values except ± 2.

3. Show that $f(x) = \left|x^2 - 4\right|$ is not differentiable at $x = -2$.

Check your answer by turning to the Appendix.

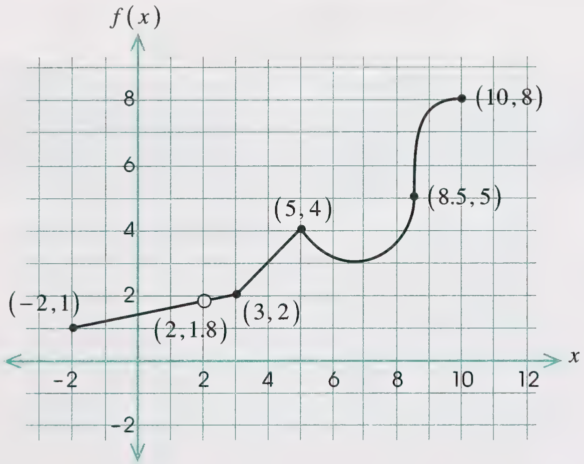
Example 4 showed you a function, $y = f(x)$, that was not differentiable at all values in its domain. This function was not a smooth curve; there were sharp transitions in the graph, namely at ± 2, where the tangent did not exist. Under what other conditions will a function fail to be differentiable at a given value of *x*? Remember, for the function to be differentiable at $x = a$, $f'(a)$ must exist.

- If the tangent is vertical at *a*, then the slope of the tangent $f'(a)$ is undefined.
- If the function is discontinuous at $x = a$, then $f(a)$ does not exist.

If $f(a)$ is undefined, then $f'(a)$ must also be undefined. After all, $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. The rule of thumb is that the domain of f' (the set of values for which *f* is differentiable) must be a subset of the domain of *f*.

Example 5

Identify the values of x for which $y = f(x)$ is not differentiable. Explain why.



Solution

The function is not differentiable at the following values:

- $x = -2$

  The left-hand limit, $\lim\limits_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h}$, does not exist. The function itself is not defined for $x < -2$.

- $x = 2$

  The function is discontinuous at $x = 2$; $f(2)$ must be defined for $f'(2)$ to exist.

- $x = 3$

  There is a sharp transition in the curve.

- $x = 5$

  There is a sharp transition in the curve.

- $x = 8.5$

  The tangent line is vertical; its slope is undefined.

- $x = 10$

  The right-hand limit, $\lim\limits_{h \to 0^+} \frac{f(10+h) - f(10)}{h}$, does not exist. The function itself is not defined for $x > 10$.

4. Sketch each function. Use a graphing calculator if necessary. State where each function is not differentiable. Explain why.

a. $f(x) = -\sqrt{x-1}$

b. $f(x) = \frac{x^2 - 4}{x - 2}$

c. $y = |x - 2|$

d. $y = \frac{1}{x}$

e. $y = x^{\frac{1}{3}}$

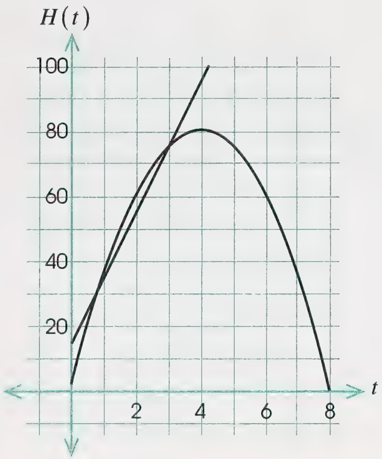
Check your answers by turning to the Appendix.

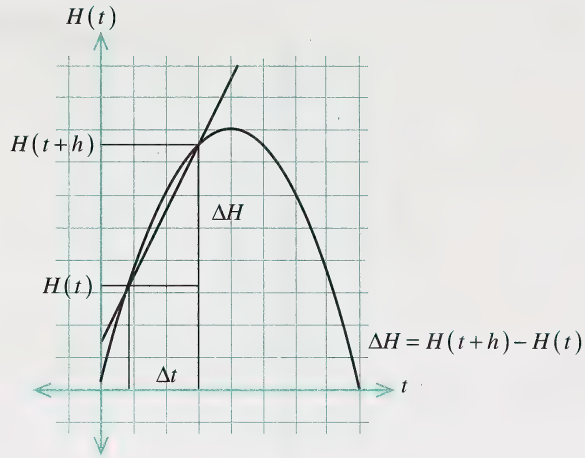
A derivative can be interpreted as a rate. Why?

Example 6

An arrow is shot straight upward at a speed of 40 m/s. Its height $H(t)$ (in metres) after time t (in seconds) is described by $H(t) = -5t^2 + 40t + 2$. Graph the function. State the meaning of a secant through the points $(t, H(t))$ and $(t+h, H(t+h))$. Find the derivative of H with respect to t. Use the derivative to find the speed of the arrow at $t = 1$.

Solution





The slope of the secant line is as follows:

$$m_{\text{sec}} = \frac{\Delta H}{\Delta t} = \frac{H(t+h) - H(t)}{h}$$

The slope of the secant is the $\frac{\text{change in height}}{\text{change in time}}$, or the average speed over the time interval in question. If $h \to 0$, the average speed approaches the speed of the arrow the instant when $\text{time} = t$.

$$\text{Instantaneous speed} = \lim_{h \to 0} \frac{H(t+h) - H(t)}{h}$$

But this is simply the derivative of H with respect to t.

Therefore, instantaneous speed = slope of the tangent = $H'(t)$.

What is the derivative?

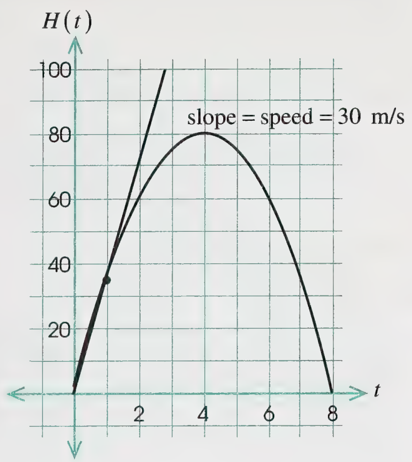
$$\begin{aligned} H(t) &= -5t^2 + 40t + 2 \\ H(t+h) &= -5(t+h)^2 + 40(t+h) + 2 \\ &= -5\left(t^2 + 2ht + h^2\right) + 40t + 40h + 2 \\ &= -5t^2 - 10ht - 5h^2 + 40t + 40h + 2 \end{aligned}$$

$$\begin{aligned} H'(t) &= \lim_{h \to 0} \frac{H(t+h) - H(t)}{h} \\ &= \lim_{h \to 0} \frac{\left(-5t^2 - 10ht - 5h^2 + 40t + 40h + 2\right) - \left(-5t^2 + 40t + 2\right)}{h} \\ &= \lim_{h \to 0} \frac{-5t^2 - 10ht - 5h^2 + 40t + 40h + 2 + 5t^2 - 40t - 2}{h} \\ &= \lim_{h \to 0} \frac{-10ht - 5h^2 + 40h}{h} \\ &= \lim_{h \to 0} (-10t + 40 - 5h) \\ &= -10t + 40 \end{aligned}$$

At $t = 1$ second, the arrow's speed is exactly $H'(1)$.

$$\begin{aligned} H'(1) &= -10(1) + 40 \\ &= 30 \text{ m/s} \end{aligned}$$

This speed or rate is the slope of the curve at $t = 1$.

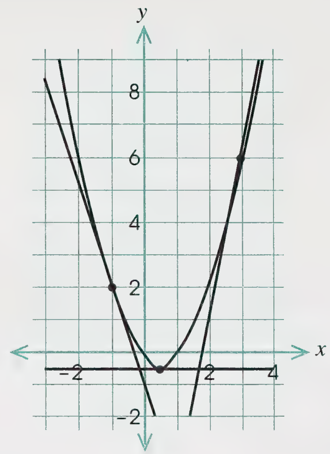


As you have just seen from Example 6, another way of looking at a derivative is as a **rate**. The derivative of a position function is speed. But how would you interpret the rate if the function is expressed in terms of x and y?

Example 7

The derivative of $y = x^2 - x$ is $\frac{dy}{dx} = 2x - 1$. Describe how y is changing with respect to x when $x = -1$, $x = 0.5$, and $x = 3$. Refer to the graph of the original function and the tangents at each point.

Solution



Now, $f'(x) = 2x - 1$.

At $x = -1$, the slope of the tangent line is $f'(-1) = 2(-1) - 1 = -3$.

Since $f(-1) = (-1)^2 - (-1) = 2$, the tangent touches the curve at $(-1, 2)$. At that point, y is decreasing as x increases. At what rate is y changing? y is decreasing by 3 when x increases by 1. This rate is an instantaneous rate of change of y with respect to x; the rate is continuously changing as you move from point to point along the curve. It is only when you move along the tangent line that y decreases by 3 when x increases by 1.

At $x = 0.5$, the slope of the tangent line is $f'(0.5) = 2(0.5) - 1 = 0$. Since $f(0.5) = (0.5)^2 - (0.5) = -0.25$, this tangent touches the curve at $(0.5, -0.25)$. At this point, because the slope is 0, y is not changing with respect to x. Because y is momentarily constant, this point is called a **stationary point**.

At $x = 3$, the slope of the tangent line is $f'(3) = 2(3) - 1 = 5$. Since $f(3) = (3)^2 - (3) = 6$, this tangent touches the curve at $(3, 6)$. At this point, because the slope is 5, y is increasing with respect to x. y increases by 5 when x increases by 1.

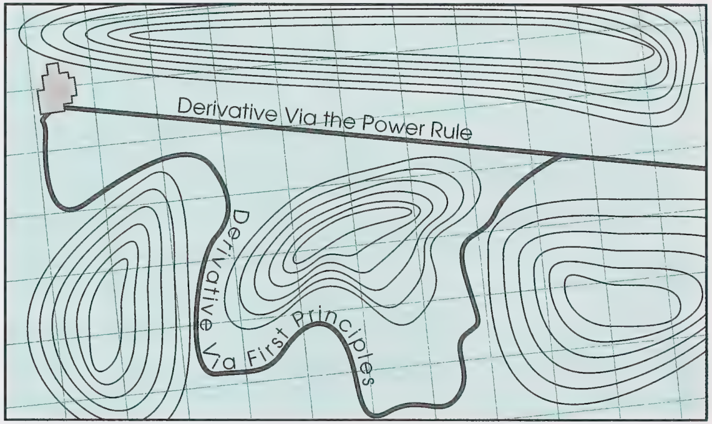
As you have seen in this activity, a derivative (which you should now be able to find from first principles) can be thought of as a slope of a tangent, as a function, and as a rate.

5. If for $y = f(x)$, $f'(x) = 3x - 6$, at what rate is y changing with respect to x at $x = 4$?

Check your answer by turning to the Appendix.

Finding derivatives is a powerful technique in analysing motion. For the arrow problem, the derivative will help you find the arrow's instantaneous speed and maximum height.

Activity 2: The Power Rule

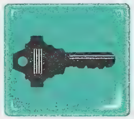


In the previous activity you were introduced to the derivative. Using the definition of the derivative, you differentiated simple polynomial functions. But suppose you wanted to find the derivative of a function such as $y = x^{10}$? As part of the process, would you have to find $(x + h)^{10}$, either by multiplying the ten factors together or perhaps by using the Binomial Theorem? Either method would be extremely time consuming. In this activity you will use a shortcut called the power rule to differentiate powers.

At this point, watch the video titled *The Derivative/ The Power Rule* from the *Catch 31* series, ACCESS Network. Review the process of finding the derivative from first principles. Make notes on finding derivatives of powers and the power rule. You may find that making notes is useful as you work through the examples and the questions in this activity. This video is available from Learning Resources Distributing Centre.

In Activity 1 of this section, you were asked to find the derivatives of various powers. The following table summarizes your results.

Function	Derivative
$y = c$, where c is a constant	$\frac{dy}{dx} = 0$
$y = x^1$	$\frac{dy}{dx} = 1$
$y = x^2$	$\frac{dy}{dx} = 2x^1$
$y = x^3$	$\frac{dy}{dx} = 3x^2$



This table suggests that for $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$. The exponent n of the original function appears as the coefficient of the derivative. The exponent $n-1$, appearing in the derivative, is one less than the exponent in the original function.

What would this mean for $y = x^{-1}$?

$$y = x^{-1}$$

Subtract 1 from the original exponent; $-1-1=-2$.

$$\frac{dy}{dx} = -1x^{-1-1}$$
$$= -1x^{-2}$$

What about $y = x^4$?

$$y = x^4$$

Subtract 1 from the original exponent; $4-1=3$.

$$\frac{dy}{dx} = 4x^{4-1}$$
$$= 4x^3$$

In the next example you will prove $\frac{d}{dx}\left(x^{-1}\right) = -1x^{-2}$.

Example 1

Differentiate $y = x^{-1}$ using first principles.

Solution

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}, \text{ where } \Delta x = h \\
&= \lim_{h \to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{1(x+h)}{x(x+h)}}{\frac{h}{1}} \\
&= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - h}{x(x+h)} \bullet \frac{1}{h} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x^2} \text{ or } -1x^{-2}
\end{aligned}$$

Is all of this simply a coincidence, or is $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ a correct statement of the **power rule**? The following proof of the power rule is for $n \in N$.

Consider $y = x^n$, where $n \in N$.

Assume when x changes by h, y changes by k; therefore, $y + k = (x+h)^n$.

Now use the Binomial Theorem to expand the expression $(x+h)^n$.

$$\begin{aligned}
y + k &= (x+h)^n \\
&= x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 \\
&\quad + \frac{n(n-1)(n-2)}{6}x^{n-3}h^3 + \cdots + h^n
\end{aligned}$$

Substitute $y = x^n$ in the preceding equation.

$$k = nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}h^3 + \cdots + h^n$$

$$= h\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \frac{n(n-1)(n-2)}{6}x^{n-3}h^2 + \cdots + h^{n-1}\right)$$

$$\frac{k}{h} = \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \frac{n(n-1)(n-2)}{6}x^{n-3}h^2 + \cdots + h^{n-1}\right)$$

Now find the limit when $h \to 0$.

$$\lim_{h\to 0}\left(\frac{k}{h}\right) = \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots h^{n-1}\right]$$

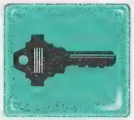
$$= \lim_{h\to 0} nx^{n-1} + \lim_{h\to 0}\frac{n(n-1)}{2}x^{n-2}h + \cdots + \lim_{h\to 0} h^{n-1}$$

$$= nx^{n-1} + 0 + \cdots + 0$$

$$= nx^{n-1}$$

Since h is the increase in x and k is the corresponding increase in y, then $h = \Delta x$ and $k = \Delta y$.

$$\therefore \lim_{h\to 0}\frac{k}{h} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \text{ and } \frac{dy}{dx} = nx^{n-1}$$



Power Rule:

$$\frac{dy}{dx} = nx^{n-1}$$

It can also be proven that if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$ for any real value of n. You will be using this result throughout this module.

Example 2

Find the derivative of x^5.

Solution

Let $y = x^5$.

$$\frac{dy}{dx} = 5x^{5-1}$$

Subtract 1.

$$= 5x^4$$

The original exponent becomes the coefficient of the derivative.

Example 3

Find the derivative of $x^{\frac{1}{2}}$.

Solution

Let $y = x^{\frac{1}{2}}$.

Bring down the original exponent as the coefficient. Subtract 1 from the original exponent to find the new exponent.

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)x^{\frac{1}{2}-1}$$

$$= \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2x^{\frac{1}{2}}}$$

Example 4

Find the derivative of x^{-4}.

Solution

Let $y = x^{-4}$.

$$\frac{dy}{dx} = (-4)x^{-4-1}$$

$$= -4x^{-5}$$

$$= \frac{-4}{x^5}$$

Rewrite the answer using positive exponents only.

1. Differentiate the following:

a. $y = x^{12}$

b. $f(x) = x^{-5}$

c. $y = \frac{1}{x^{10}}$

d. $y = \sqrt[3]{x}$

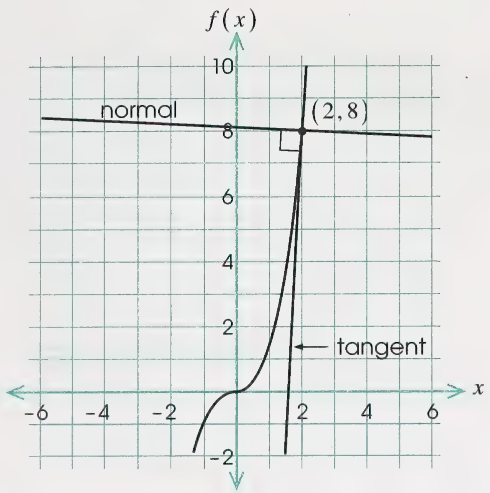
Check your answers by turning to the Appendix.

Next, you will apply the power rule in problem-solving situations.

Example 5

Find the equations of the tangent and the normal to $f(x) = x^3$ at $x = 2$.

Solution



First, find the slope of the tangent line. Since $f(2) = 8$, the point of contact of the tangent is $(2, 8)$.

Next, find the slope of the tangent at $x = 2$.

$$f'(x) = 3x^2$$
$$f'(2) = 3(2)^2$$
$$= 12$$

Tangent Calculations

Method 1

Use the formula $y - y_1 = m(x - x_1)$ to find the linear equation.

$$y - 8 = 12(x - 2), \text{ where } m = 12 \text{ and } (x_1, y_1) = (2, 8)$$
$$y - 8 = 12x - 24$$
$$12x - y - 16 = 0$$

Method 2

Use the formula $y = mx + b$ to find the linear equation.

$$y = 12x + b, \text{ where } m = 12$$

To evaluate b, substitute $(2, 8)$ into the equation.

$$8 = 12(2) + b$$
$$b = -16$$

$$\therefore\ y = 12x - 16$$
$$12x - y - 16 = 0$$

The equation of the tangent is $12x - y - 16 = 0$.

Normal Calculations

Since the slope of the tangent is 12, the slope of the normal is $-\frac{1}{12}$.

Use $y = mx + b$.

$\therefore y = -\frac{1}{12}x + b$, where $m = -\frac{1}{12}$.

To evaluate b, substitute $(2, 8)$ into $y = mx + b$.

$$\begin{aligned} 8 &= -\frac{1}{12}(2) + b \\ 96 &= -2 + 12b \\ b &= \frac{98}{12} \end{aligned}$$

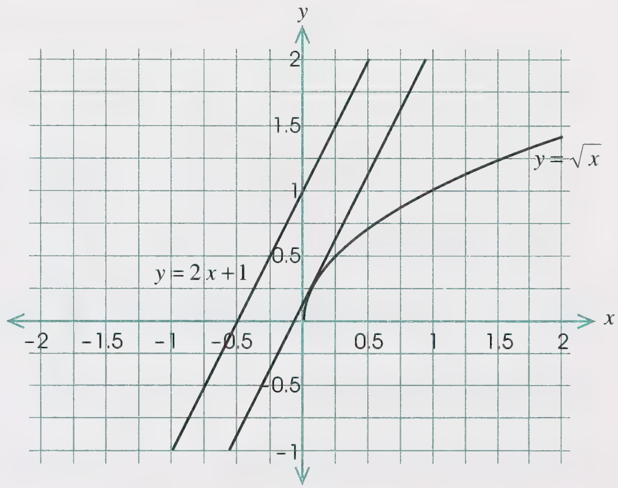
$$\begin{aligned} \therefore\ y &= -\frac{1}{12}x + \frac{98}{12} \\ 12y &= -x + 98 \\ x + 12y - 98 &= 0 \end{aligned}$$

The equation of the normal is $x + 12y - 98 = 0$.

Example 6

Find the point on the graph of $y = \sqrt{x}$ that has a tangent parallel to $y = 2x + 1$.

Solution



The slope of $y = 2x + 1$ is the coefficient of x. The slope is 2.

The derivative of $y = \sqrt{x}$ or $y = x^{\frac{1}{2}}$ is

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \end{aligned}$$

Since the derivative is the slope of the tangent line, equate the derivative to the slope of the given line.

$$\frac{1}{2}x^{-\frac{1}{2}} = 2$$
$$(2)\frac{1}{2}x^{-\frac{1}{2}} = 2(2)$$
$$x^{-\frac{1}{2}} = 4$$
$$x^{\frac{1}{2}} = \frac{1}{4} \quad \text{(Take the reciprocal of both sides.)}$$
$$\left(x^{\frac{1}{2}}\right)^2 = \frac{1^2}{4^2}$$
$$x = \frac{1}{16}$$
$$y = \sqrt{\frac{1}{16}}$$
$$= \frac{1}{4}$$

The point of contact of the tangent is $\left(\frac{1}{16}, \frac{1}{4}\right)$.

2. Determine the point on the graph of $y = x^2$ where the normal has a slope of $\frac{1}{2}$.

3. Find the equation(s) of the normal(s) to $y = x^5$ if the slope of the normal(s) is $-\frac{1}{5}$.

Check your answers by turning to the Appendix.

Do you agree that using the power rule to find derivatives is much faster than using first principles?

Activity 3: The Derivative of $c \bullet f(x)$

Have you ever driven up a mountain road and wondered if your car would negotiate the steep grades? How would the slope of that road change if your destination were twice as high? three times as high?

When using the power rule, you differentiated functions such as $y = x^2$, $y = x^{10}$, and $y = x^{\frac{1}{2}}$, and found the slopes of the graphs at various points.

But what would you do if you were given functions such as $y = 3x^2$, $y = \frac{2}{5}x^{10}$, or $y = \sqrt{2}\,x^{\frac{1}{2}}$, and would the slopes of the graphs be affected by the coefficients?

Suppose $y = cf(x)$, where c is a constant.

$$y + \Delta y = cf(x+h)$$
$$\therefore\ \Delta y = cf(x+h) - y$$
$$\doteq cf(x+h) - cf(x)$$

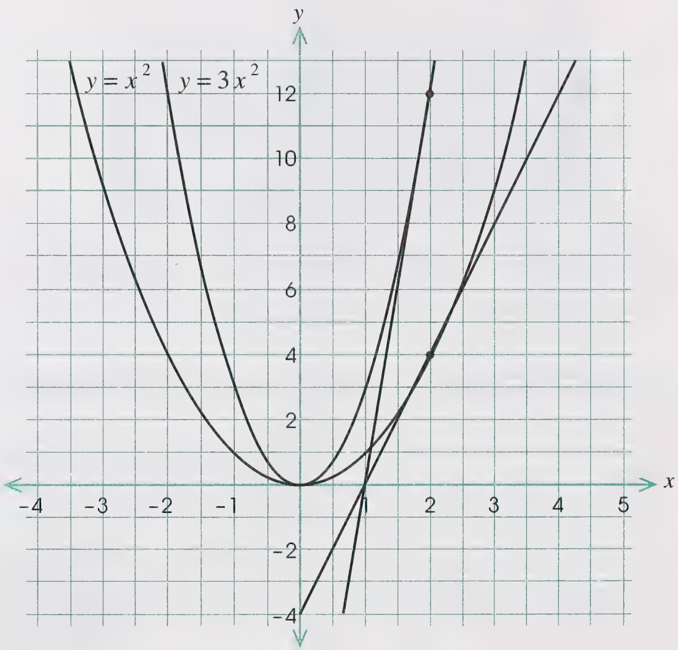
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}, \text{ where } \Delta x = h$$
$$= \lim_{h \to 0} \frac{c[f(x+h) - f(x)]}{h}$$
$$= \lim_{h \to 0} c \bullet \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(The limit of a product is the product of the limits.)}$$
$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(The limit of a constant is the constant itself.)}$$
$$= cf'(x)$$

The preceding analysis proves that the derivative of the product of a constant and a function is the derivative of the function multiplied by the constant. That is, $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$.

Example 1

Compare the graphs of $f_1(x) = x^2$ and $f_2(x) = 3x^2$. Determine the slope of each graph at $x = -1$, $x = 0$, $x = 1$, $x = 2$, and $x = 3$. How are the slopes related? Explain your results using the graphs at $x = 2$.

Solution



The graph of $f_2(x) = 3x^2$ is the graph of $f_1(x) = x^2$ stretched by a factor of 3, because $f_2(x) = 3f_1(x)$. Each point on the graph of $f_2(x) = 3x^2$ lies three times as far from the *x*-axis as does the point on the graph of $f_1(x) = x^2$ corresponding to the same value of *x*. For instance, $f_1(2) = 4$, whereas $f_2(2) = 3(2^2) = 12$.

Now, how are the slopes of the two graphs related?

$$f_1(x) = x^2 \text{ and } f_1'(x) = 2x$$

$$\begin{aligned} f_2(x) = 3x^2 \text{ and } f_2'(x) &= 3(2x) \\ &= 6x \end{aligned}$$

Look at the following table to compare the slopes of the tangents.

x	−1	0	1	2	3
$f_1'(x)$	−2	0	2	4	6
$f_2'(x)$	−6	0	6	12	18

The slope of a tangent $y = 3x^2$ is three times the slope of a tangent to $y = x$ for the same value of *x*. *y* is changing three times as fast with respect to *x* along the graph of $f_2(x) = 3x^2$ as it is along the graph of $f_1(x) = x^2$. In particular, when $x = 2$, the slope of the tangent to $f_1(x) = x^2$ is 4. When $x = 2$, the slope of the tangent to $f_3(x) = 3x^2$ is 12, three times as large.

Not only is the graph of f_2 stretched by a factor of 3, its slope is increased threefold.

Example 2

Differentiate $f(x) = 3x^{-\frac{1}{3}}$, $y = 5x^4$, and $y = -3\sqrt{x}$.

Solution

$$\begin{aligned} f'(x) &= 3x^{-\frac{1}{3}} \\ &= 3\left(-\frac{1}{3}\right)x^{-\frac{1}{3}-1} \\ &= -x^{-\frac{4}{3}} \end{aligned}$$

$$\begin{aligned} y &= 5x^4 \\ \frac{dy}{dx} &= 5\left(4x^{4-1}\right) \\ &= 20x^3 \end{aligned}$$

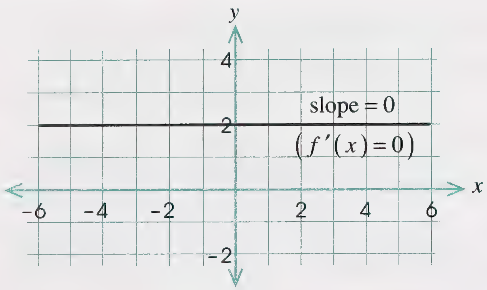
$$\begin{aligned} y &= -3\sqrt{x} \\ &= -3x^{\frac{1}{2}} \\ y' &= -3\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} \\ &= -\frac{3}{2}x^{-\frac{1}{2}} \end{aligned}$$

Example 3

Differentiate $f(x) = 2$ and $y = 3x$. Use graphs to interpret your results.

Solution

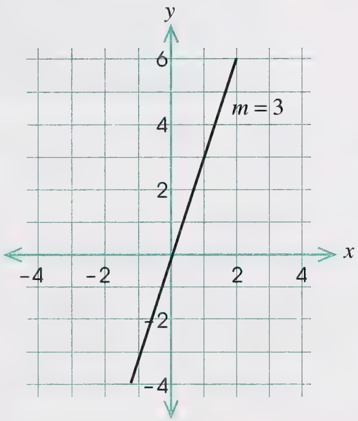
Since $f(x) = 2$ is a constant function, $f'(x) = 0$.



The graph is a horizontal line; its slope is 0. The derivative is the slope of the curve.

$y = 3x$

$\frac{dy}{dx} = 3$



The graph of $y = 3x$ is an oblique line. The derivative is simply the slope, 3.

Remember: The derivative of a constant is 0.

If c is a constant, $\frac{d}{dx}(c) = 0$.

If the function is linear, its derivative is its slope.

If $y = mx$, $\frac{d}{dx}(mx) = m$.

Now it's your turn.

1. Differentiate the following:

a. $y = -7x$ **b.** $y = \frac{x^6}{4}$ **c.** $y = -\frac{2}{x^2}$

d. $x^3 y = 3$ **e.** $y = \frac{3}{\sqrt{x}}$ **f.** $y = \frac{1}{3}$

2. Graph $3x^3 y = 1$ and find the point(s) where the slope is -1.

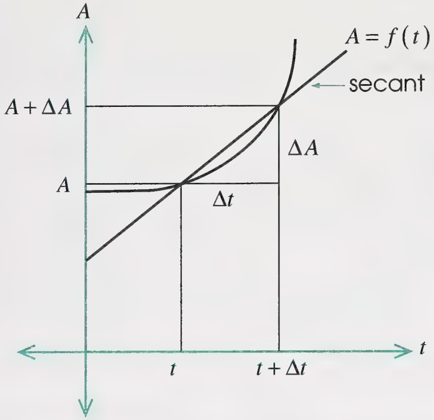
Check your answers by turning to the Appendix.

What makes one mountain road more difficult to negotiate than another is often slope or gradient. The role of the constant c in the function $y = c \bullet f(x)$ directly affects the slope of the curve.

Activity 4: Sums and Differences

If you have painted your home, you probably have appreciated the assistance of another person. Two can work faster than one!

If you were working alone, the area A that you would paint is a function of time t. The function $A = f(t)$ could be graphed to show your progress.

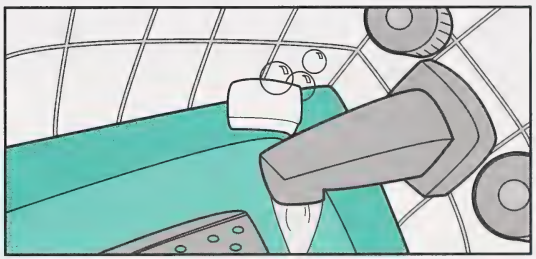


For a given time interval Δt, the additional area covered could be represented by ΔA. The ratio $\frac{\Delta A}{\Delta t}$ is the average rate at which the surface was painted. For instance, if you painted $50\ \text{m}^2$ in twenty minutes, you would have averaged $\frac{50\ \text{m}^2}{20\ \text{min}}$ or $2.5\ \text{m}^2/\text{min}$. This ratio is the slope of the secant line in the diagram. Of course, the rate at which you paint varies from moment to moment. To find your rate at a particular instant of time, you would have to find $\lim\limits_{\Delta t \to 0} \Delta A$. This is simply the derivative, $\frac{dA}{dt}$ of $A = f(t)$.

You will recall that one of the interpretations of a derivative is a rate. Now, if your "painting function" is $A = f_1(t)$ and your friend's is $A = f_2(t)$, then the area covered working together in time t would be $A = f_1(t) + f_2(t)$. It seems reasonable that if you were to differentiate this sum, the result should be $\frac{dA}{dt} = \frac{d}{dt} f_1(t) + \frac{d}{dt} f_2(t)$. This derivative says that the combined rate is your rate added to your friend's rate. If you were painting at $2.5\ \text{m}^2/\text{min}$, and your friend at $3\ \text{m}^2/\text{min}$, then the walls would be covered at a rate of $5.5\ \text{m}^2/\text{min}$.

This result seems reasonable in a variety of contexts. Rates are commonly added or subtracted. For instance, a passenger in an aircraft is walking towards the cabin at the front of the plane at a speed of 2 m/s, and the plane is travelling at 100 m/s. Therefore, the net speed of the passenger is 102 m/s. If the passenger were to walk towards the tail of the plane, the net speed would be $100\ \text{m/s} - 2\ \text{m/s} = 98\ \text{m/s}$.

If you were filling your bathtub at twenty litres per minute, and if the plug worked itself loose, releasing water at fifteen litres per minute, the net result would be the tub being filled at $20\ \text{L/min} - 15\ \text{L/min} = 5\ \text{L/min}$.



The preceding discussion suggests that if $y = f(x) \pm g(x)$, then, $\frac{dy}{dx} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$. This is the **sum rule** or **difference rule**.

To prove this statement, assume that if x is changed by h to $x+h$, then $f(x)$ becomes $f(x+h)$ with the change in f equal to $f(x+h) - f(x)$, and $g(x)$ becomes $g(x+h)$ with the change in g equal to $g(x+h) - g(x)$. Now, the change in y equals the change in f plus or minus the change in g.

Symbolically, $\Delta y = \Delta f \pm \Delta g$.

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x} \pm \frac{\Delta g}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta h \to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{\Delta h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$\frac{dy}{dx} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

where $\Delta x = h$

At this time, watch the segment dealing with sums and differences from the video *The Sum Rule and the Chain Rule for Derivatives* from the *Catch 31* series, ACCESS Network. In particular, make notes on how the sum rule is applied. This video is available from Learning Resources Distributing Centre.

Example 1

Differentiate $y = 3x^4 - 2x^3 + 5x^2 - x + 7$.

Solution

$$\frac{dy}{dx} = \frac{d(3x^4)}{dx} - \frac{d(2x^3)}{dx} + \frac{d(5x^2)}{dx} - \frac{d(x)}{dx} + \frac{d(7)}{dx}$$
$$= 12x^3 - 6x^2 + 10x - 1$$

Example 2

Find the slope of the tangent to $f(x) = 3x^2 - 5x + 4$ at $x = 4$.

Solution

First find the derivative.

$$f'(x) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(4)$$
$$= 3(2x) - 5(1) + 0$$
$$= 6x - 5$$

The slope of the tangent is the value of the derivative at $x = 4$.

$$f'(4) = 6(4) - 5$$
$$= 24 - 5$$
$$= 19$$

The slope of the tangent is 19.

1. Differentiate each of the following:

   a. $y = 2x^4 - 3x^3 - 5x + 9$
   b. $y = x^5 - 8x^4 + 3x^3$
   c. $y = 5x^4 - \frac{1}{2}x^3 + 7$
   d. $y = \frac{1}{2}x^3 - 2x^2 + x - 5$

2. Find the slope of the tangent to $f(x) = x^2 - x + 1$ at $x = 1$.

Check your answers by turning to the Appendix.

The problem of finding the derivative of a sum or difference is a problem in adding or subtracting rates, just as the rate at which a house can be painted changes when others assist with the task.

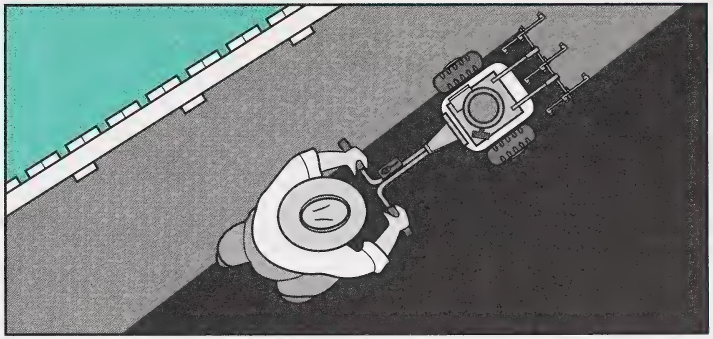
Activity 5: The Product Rule

You should now be able to find the derivative of a function such as $y = 3x^{\frac{1}{2}} - 4x^3$. But, could you find the derivative of a product like $y = (2x-1)(3x-4)$?

One approach is to expand, to multiply the binomials together, and then differentiate.

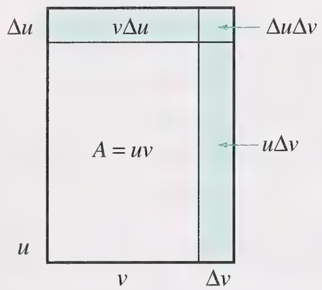
$$y = 6x^2 - 8x - 3x + 4$$
$$y = 6x^2 - 11x + 4$$
$$\frac{dy}{dx} = 6(2x) - 11$$
$$= 12x - 11$$

However, there is an alternate method.



Suppose a farmer is tilling a rectangular garden by starting from the centre. As the farmer works around the perimeter, the area A of the tilled rectangular portion can be found by multipling its length u by its width v $(A = uv)$.

As the farmer tills along the length, the width changes by Δv. As the farmer tills along the width, the length changes by Δu. The changes to the area ΔA are shown in the following diagram.



The change in area as illustrated in the diagram is $u\Delta v + v\Delta u + \Delta u\Delta v$. You can use this result to develop a rule for differentiating a product. Consider *u* and *v* as functions of *x*; that is, $u = f(x)$ and $v = g(x)$. Let *y* be the product of these functions, $y = uv$. Assume the derivatives exist.

If *x* changes by Δx, then *u* changes by Δu, *v* changes by Δv, and *y* changes by Δy.

But, as you can see from the diagram, $\Delta y = u\Delta v + v\Delta u + \Delta v\Delta u$.

Next, divide each term by Δx to obtain $\frac{\Delta y}{\Delta x}$.

$$\frac{\Delta y}{\Delta x} = \frac{u\Delta v}{\Delta x} + \frac{v\Delta u}{\Delta x} + \frac{\Delta v\Delta u}{\Delta x}$$

Multiply the third fraction on the right side by $\frac{\Delta x}{\Delta x}$ to obtain $\frac{\Delta u}{\Delta x}$.

$$\frac{\Delta y}{\Delta x} = \frac{u\Delta v}{\Delta x} + \frac{v\Delta u}{\Delta x} + \frac{\Delta v\Delta u}{\Delta x\Delta x}\Delta x$$

Take the limit as $\Delta x \to 0$.

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{u\Delta v}{\Delta x} + \lim_{\Delta x\to 0}\frac{v\Delta u}{\Delta x} + \lim_{\Delta x\to 0}\frac{\Delta v\Delta u}{\Delta x\Delta x}\Delta x$$

Remember, the limit of a product is the product of a limit!

$$\begin{aligned}\therefore\ \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0} u \bullet \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} + \lim_{\Delta x\to 0} v \bullet \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\\ &\quad + \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}\bullet\frac{\Delta u}{\Delta x}\bullet\Delta x\\ &= u \bullet \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} + v \bullet \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\\ &\quad + \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}\bullet\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\bullet\lim_{\Delta x\to 0}\Delta x\\ \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} + \frac{dv}{dx}\bullet\frac{du}{dx}(0)\\ \frac{d}{dx}(uv) &= u\frac{dv}{dx} + v\frac{du}{dx},\ \text{where } y = uv\end{aligned}$$

This is called the **product rule**. To find the derivative of a product, multiply the first factor by the derivative of the second factor; then add the second factor multiplied by the derivative of the first.

Now you will apply this rule.

Example 1

Differentiate $y=(2x-3)(5x-1)$.

Solution

Method 1

Multiply the factors, and differentiate the product.

$$\begin{aligned} y &= 10x^2-2x-15x+3 \\ &= 10x^2-17x+3 \end{aligned} \qquad \begin{aligned} \frac{dy}{dx} &= 10(2x)-17(1)+0 \\ &= 20x-17 \end{aligned}$$

Method 2

Use the product rule. Let $u=2x-3$ and $v=5x-1$.

$$\therefore \frac{du}{dx}=2 \text{ and } \frac{dv}{dx}=5$$

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\begin{aligned} \frac{dy}{dx} &= (2x-3)(5)+(5x-1)(2) \\ &= 10x-15+10x-2 \\ &= 20x-17 \end{aligned}$$

Even though both methods yield the same results, using the product rule in most instances is faster. In the following examples, the derivatives will be obtained using an abbreviated form of the product rule.

Example 2

Find the derivative of $(x-1)(x+2)$.

Solution

Let $y=(x-1)(x+2)$.

$$\begin{aligned} \frac{dy}{dx} &= (x-1)\frac{d}{dx}(x+2)+(x+2)\frac{d}{dx}(x-1) \\ &= (x-1)(1)+(x+2)(1) \\ &= (x-1)+(x+2) \\ &= 2x+1 \end{aligned}$$

Watch the first segment of the video *The Product Rule and the Quotient Rule* from the *Catch 31* series, ACCESS Network. This segment deals with the development and application of the product rule. This video is available from Learning Resources Distributing Centre.

Example 3

Find the derivative of $\left(x^3+2x^2-3x+5\right)\left(x^2-x+8\right)$.

Solution

Let $y=\left(x^3+2x^2-3x+5\right)\left(x^2-x+8\right)$.

$$\begin{aligned}\frac{dy}{dx} &= \left(x^3+2x^2-3x+5\right)\frac{d}{dx}\left(x^2-x+8\right) \\ &\quad +\left(x^2-x+8\right)\frac{d}{dx}\left(x^3+2x^2-3x+5\right) \\ &= \left(x^3+2x^2-3x+5\right)(2x-1)+\left(x^2-x+8\right)\left(3x^2+4x-3\right) \\ &= 2x^4+4x^3-6x^2+10x-x^3-2x^2+3x-5+3x^4+4x^3 \\ &\quad -3x^2-3x^3-4x^2+3x+24x^2+32x-24 \\ &= 5x^4+4x^3+9x^2+48x-29\end{aligned}$$

1. Differentiate each of the following:

a. $y=\left(x^2-3x+4\right)\left(2x^2-x+1\right)$

b. $y=\left(x^2+4x-5\right)\left(3x^2+x-1\right)$

c. $y=x^{-2}(x+3)$

d. $y=x^{-5}(x-7)$

2. Differentiate $y=(x+1)(x-5)$ by first using the product rule. Then, expand and differentiate. Compare your answers.

Check your answers by turning to the Appendix.

You may wish to review the introduction to this activity to see how the product rule was modelled using the area of a tilled garden.

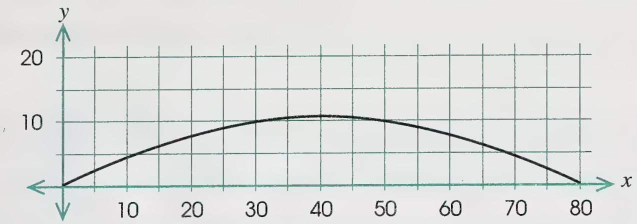
Activity 6: The Chain Rule

A softball player is pitching a ball. How high will it go? How far will it go?

If the pitcher releases the ball at a speed of 30 m/s and at an angle of 30° to the horizontal, the height h (in metres) of the ball at any time t (in seconds) would be approximated by $y=15t-5t^2$. Its distance x (in metres) from the pitcher (measured along the ground) is $x=26t$.

The trajectory of the ball can be plotted as a function in x and y. Simply solve $x=26t$ for t; then substitute the value of t into $y=15t-5t^2$.

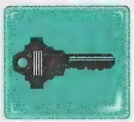
$$\text{At } t=\frac{x}{26},\ y=15\left(\frac{x}{26}\right)-5\left(\frac{x}{26}\right)^2$$
$$=\frac{15}{26}x-\frac{5}{676}x^2$$



You can find $\frac{dy}{dt}$, the rate at which the ball rises. You can also find $\frac{dx}{dt}$, the ball's horizontal speed. You can even find $\frac{dy}{dx}$, the rate of change of y (height) with respect to x (horizontal displacement). But how are these three rates related?

Knowing this relationship will help you differentiate a function such as $y=(2x-1)^{10}$. Certainly you could find the answer by first multiplying the ten factors of $(2x-1)$ together. Imagine how long this would take! You could also change the function to $y=u^{10}$ by replacing $(2x-1)$ by u. Immediately, $\frac{dy}{du}=10u^9$. But you want $\frac{dy}{dx}$. How are $\frac{dy}{du}$ and $\frac{dy}{dx}$ related?

The link between these derivatives involves composite functions and the chain rule.



Chain Rule:

$$\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$$

The development of the chain rule $\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$ follows:

- Let u be a differentiable function of x; that is, $u=f(x)$ and $\frac{du}{dx}$ exists.
- Let y be a differentiable function of u; that is, $y=g(u)$ and $\frac{dy}{du}$ exists.
- The functions may combined as follows:

$$y=g(u)$$

Replace u by $f(x)$.

$$y=g(f(x))$$

Your task is to find $\frac{dy}{dx}$.

Now, when x increases by a small amount, u increases by a small amount; when u increases by a small amount, y increases by a small amount as well.

If $\Delta x\to 0$, then $\Delta u\to 0$ and $\Delta y\to 0$.

Also, $\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta x}$

$$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta u}\bullet\frac{\Delta u}{\Delta x}$$

Next, take the limit as $x\to 0$.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \bullet \frac{\Delta u}{\Delta x}$$

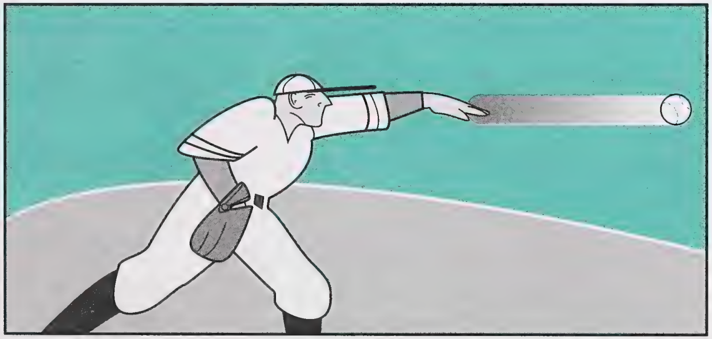
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \bullet \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \bullet \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \qquad (\text{since } \Delta u \to 0 \text{ as } \Delta x \to 0)$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

Occasionally, you will see the chain rule written as $\frac{dy}{du} = \frac{\frac{dy}{dx}}{\frac{du}{dx}}$. Now you will apply the chain rule to the problem outlined at the beginning of this activity.

Example 1



A ball is hurled at a speed of 30 m/s and at an angle of 30° to the horizontal. The height h (in metres) of the ball at any time t (in seconds) is approximated by $y = 15t - 5t^2$. Its distance x (in metres) from you, measured along the ground, is $x = 26t$. Find $\frac{dy}{dx}$ using the chain rule.

Solution

Since $y = 15t - 5t^2$, $\frac{dy}{dt} = 15 - 10t$. This is the vertical speed at time t. Also, because $x = 26t$, $\frac{dx}{dt} = 26$. This is the horizontal speed.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{15 - 10t}{26}$$

Express the derivative in terms of x.

If $x = 26t$, then $t = \frac{x}{26}$.

This may be confirmed by differentiating the function $y=\frac{15}{26}x-\frac{5}{676}x^2$, which you will recall was the composition function for this problem.

$$\begin{aligned}\frac{dy}{dx}&=\frac{15}{26}(1)-\frac{5(2x)}{676}\\&=\frac{15}{26}-\frac{10x}{676}\end{aligned}$$

Example 2

Differentiate $y=(2x-1)^{10}$.

Solution

Let $u=2x-1$; then, $\frac{du}{dx}=2$.

If $y=u^{10}$, then $\frac{dy}{du}=10u^9$.

Link these derivatives using the chain rule.

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\bullet\frac{du}{dx}\\&=10u^9(2)\\&=20u^9\\&=20(2x-1)^9\qquad(\text{since } u=2x-1)\end{aligned}$$

Example 3

Differentiate $y=\left(x^2-2x\right)^5$.

Solution

Let $u=x^2-2x$; thus, $\frac{du}{dx}=2x-2$.

$$\begin{aligned}y&=u^5\\\frac{dy}{du}&=5u^4\\\frac{dy}{dx}&=\frac{dy}{du}\bullet\frac{du}{dx}\\&=5u^4(2x-2)\\&=5\left(x^2-2x\right)^4(2x-2)\end{aligned}$$

Example 4

Differentiate $y=\frac{1}{3-x}$.

Solution

Let $u=3-x$; thus, $\frac{du}{dx}=(-1)$.

$$y = \frac{1}{u} = u^{-1}$$

$$\frac{dy}{du} = -u^{-2}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \bullet \frac{du}{dx} \\ &= -u^{-2}(-1) \\ &= u^{-2} \\ &= \frac{1}{u^2} \\ &= \frac{1}{(3-x)^2}\end{aligned}$$

Example 5

Differentiate $y = \frac{-3}{\sqrt{5x+3}}$.

Solution

Let $u = (5x+3)$.

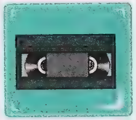
$$D_x u = 5$$

$$\begin{aligned}y &= \frac{-3}{u^{\frac{1}{2}}} \\ &= -3u^{-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}D_u y &= (-3)\left(-\frac{1}{2}\right)u^{-\frac{3}{2}} \\ &= \frac{3}{2}u^{-\frac{3}{2}}\end{aligned}$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\begin{aligned}\therefore\ D_x y &= \frac{3}{2}u^{-\frac{3}{2}} \bullet (5) \\ &= \frac{15}{2}u^{-\frac{3}{2}} \\ &= \frac{15}{2}(5x+3)^{-\frac{3}{2}}\end{aligned}$$



Before doing the following questions, watch the segment on the chain rule from the video *The Sum Rule and the Chain Rule for Derivatives* from the *Catch 31* series, ACCESS Network. Pay particular attention to the algebraic techniques for finding a derivative using the chain rule. This video is available from Learning Resources Distributing Centre.

1. Differentiate each function.

a. $y = (5x-3)^6$

b. $y = (6x+7)^7$

c. $y = \frac{1}{x-5}$

d. $y = \frac{4}{x^2+2}$

e. $y = \sqrt{3x-1}$

f. $y = \sqrt{4x+3}$

g. $y = \dfrac{3}{\sqrt{x-2}}$

h. $y = \dfrac{5}{\sqrt{2x+1}}$

i. $y = \left(20 - 7x^2\right)^5$

j. $y = \dfrac{3}{x^{\frac{5}{3}}}$

2. A balloon is being inflated. Its radius r is increasing at a rate of 4 cm/s; that is, $\frac{dr}{dt} = 4$ cm/s. Use the chain rule to determine how fast the circumference C of the balloon is changing. Use $C = 2\pi r$.

Check your answers by turning to the Appendix.

You have probably noticed that if the function is of the form $y = (f(x))^n$, it can be differentiated quickly.

Let $u = f(x)$; then $\dfrac{du}{dx} = f'(x)$.

Also $y = u^n$.

$$\therefore \frac{dy}{du} = nu^{n-1}$$

From the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \bullet \frac{du}{dx} \quad \text{or} \quad y' = n(f(x))^{n-1} f'(x)\\ &= nu^{n-1}\frac{du}{dx}\end{aligned}$$

Now, try this variation of the chain rule.

Example 6

Differentiate $y = \left(3x^2 - x\right)^7$.

Solution

Let $u = 3x^2 - x$.

$$\begin{aligned}\therefore \frac{du}{dx} &= 3(2x) - 1\\ &= 6x - 1\end{aligned}$$

Using the formula $\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$,

$$\begin{aligned}y' &= 7\left(3x^2 - x\right)^{7-1}(6x-1)\\ &= 7\left(3x^2 - x\right)^6(6x-1)\end{aligned}$$

Example 7

Find the derivative of $y=\frac{3}{\sqrt{2x-1}}$.

Solution

Rewrite the radical as a power.

$$y=3(2x-1)^{-\frac{1}{2}}$$

Let $u=2x-1$.

$$\therefore \frac{du}{dx}=2$$

Using the formula $y'=nu^{n-1}\frac{du}{dx}$,

$$\begin{aligned} y' &= (3)\left(-\frac{1}{2}\right)(2x-1)^{-\frac{1}{2}-1}(2) \\ &= -3(2x-1)^{-\frac{3}{2}} \end{aligned}$$

3. Which method do you prefer? Use the previous approach to differentiate each of the following functions.

   a. $y=(3x-1)^8$

   b. $y=\frac{1}{(2x-1)^{10}}$

   c. $y=\sqrt{4-x}$

Check your answers by turning to the Appendix.

Next, you will use the chain rule together with the product rule. You will notice in the following examples that the answers are expressed in factored form. This form makes it easier to determine stationary points on the graph where the derivative is equal to zero.

Example 8

Find the derivative of $y=(2x-1)^{10}(x-1)^3$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= (2x-1)^{10}\frac{d}{dx}(x-1)^3+(x-1)^3\frac{d}{dx}(2x-1)^{10} \\ &= (2x-1)^{10}(3)(x-1)^2(1)+(x-1)^3(10)(2x-1)^9(2) \\ &= 3(2x-1)^{10}(x-1)^2+20(x-1)^3(2x-1)^9 \end{aligned}$$

Remove $(2x-1)^9(x-1)^2$ as a common factor.

$$=(2x-1)^9(x-1)^2[3(2x-1)+20(x-1)]$$

Remove the parentheses within the square brackets; then combine similar terms.

$$=(2x-1)^9(x-1)^2[6x-3+20x-20]$$
$$=(2x-1)^9(x-1)^2(26x-23)$$

Example 9

Find the derivative of $(x-3)^{-2}(x+1)$.

Solution

Let $y=(x-3)^{-2}(x+1)$.

$$\frac{dy}{dx}=(x-3)^{-2}\frac{d}{dx}(x+1)+(x+1)\frac{d}{dx}(x-3)^{-2}$$
$$=(x-3)^{-2}(1)+(x+1)(-2)(x-3)^{-3}(1)$$
$$=(x-3)^{-2}-2(x+1)(x-3)^{-3}$$
$$=(x-3)^{-3}\left[(x-3)^{1}-2(x+1)\right]$$

$(x-3)^{-3}$ is a common factor.

$$=(x-3)^{-3}[x-3-2x-2]$$

Remove the inner parentheses.

$$=(x-3)^{-3}(-x-5)$$

Combine similar terms.

Example 10

Find the derivative of $(x-1)\sqrt{x+3}$.

Solution

Let $y=(x-1)\sqrt{x+3}$.

$$\frac{dy}{dx}=(x-1)\frac{d}{dx}\sqrt{x+3}+\sqrt{x+3}\frac{d}{dx}(x-1)$$
$$=(x-1)\frac{d}{dx}(x+3)^{\frac{1}{2}}+\sqrt{x+3}\frac{d}{dx}(x-1)$$
$$=(x-1)\frac{1}{2}(x+3)^{-\frac{1}{2}}(1)+\sqrt{x+3}(1)$$
$$=\frac{1}{2}(x-1)(x+3)^{-\frac{1}{2}}+(x+3)^{\frac{1}{2}}$$
$$=\frac{1}{2}(x+3)^{-\frac{1}{2}}[(x-1)+2(x+3)]$$
$$=\frac{1}{2}(x+3)^{-\frac{1}{2}}(3x+5)$$

4. Differentiate the following.

a. $y=(x-3)^{-2}(x+1)$

b. $y=(x+4)^{-3}(x-2)$

c. $y=(x-5)\sqrt{x+2}$

d. $y=\sqrt{x-3}(x+1)$

e. $y=\sqrt{x-2}\sqrt{x+1}$

f. $y=\sqrt{x-3}\sqrt{x-5}$

Check your answers by turning to the Appendix.

The height and range of the ball mentioned at the beginning of this activity were both functions of time. When two quantities are expressed in terms of a third, the chain rule is often used to find derivatives.

Activity 7: The Quotient Rule

A function such as $y=\frac{3x-1}{x-2}$ may be differentiated by first rewriting it as $y=(3x-1)(x-2)^{-1}$ and using the product rule. However, there is a rule for the derivative of a quotient, which, in a number of instances, is more convenient to use. The rule for the derivative of a quotient is derived from the product rule.

Assume $u=f(x)$ and $v=g(x)$ are differentiable functions, and that you will be finding the derivative at a value of x for which $v\neq 0$.

If $y=\frac{u}{v}$, then $y=uv^{-1}$.

$$\begin{aligned}\frac{dy}{dx}&=u\bullet\frac{d}{dx}\left(v^{-1}\right)+v^{-1}\bullet\frac{d}{dx}(u) \qquad \text{(from the chain rule)}\\ &=u(-1)\left(v^{-1-1}\right)\frac{d}{dx}(v)+v^{-1}\frac{du}{dx}\\ &=-uv^{-2}\frac{dv}{dx}+v^{-1}\frac{du}{dx}\\ &=v^{-2}\left(-u\frac{dv}{dx}+v\frac{du}{dx}\right)\\ &=v^{-2}\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\\ &=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\end{aligned}$$

This is called the **quotient rule.**

An alternate form is $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{vu'-uv'}{v^2}$.

Notice that in the examples which follow, the derivatives are expressed in factored form. Factored form will be used to obtain critical points on the graph of functions.

Example 1

Find the derivative for $y=\frac{3x-2}{x^2+1}$.

Solution

$$\frac{dy}{dx}=\frac{\left(x^2+1\right)\frac{d}{dx}(3x-2)-(3x-2)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$=\frac{\left(x^2+1\right)(3)-(3x-2)(2x)}{\left(x^2+1\right)^2}$$

$$=\frac{3x^2+3-6x^2+4x}{\left(x^2+1\right)^2}$$

$$=\frac{-3x^2+4x+3}{\left(x^2+1\right)^2}$$

Example 2

Differentiate $y=\frac{3-x}{\sqrt{x^2-2x}}$.

Solution

$$\frac{dy}{dx}=\frac{\sqrt{x^2-2x}\,\frac{d}{dx}(3-x)-(3-x)\frac{d}{dx}\left(\sqrt{x^2-2x}\right)}{x^2-2x}$$

$$=\frac{\left(x^2-2x\right)^{\frac{1}{2}}(-1)-(3-x)\left(\frac{1}{2}\right)\left(x^2-2x\right)^{-\frac{1}{2}}(2x-2)}{\left(x^2-2x\right)}$$

$$=\frac{-\left(x^2-2x\right)^{\frac{1}{2}}-\frac{1}{2}\left(x^2-2x\right)^{-\frac{1}{2}}(3-x)(2x-2)}{\left(x^2-2x\right)}$$

$$=\frac{-\frac{1}{2}\left(x^2-2x\right)^{-\frac{1}{2}}\left[2\left(x^2-2x\right)+(3-x)(2x-2)\right]}{\left(x^2-2x\right)}$$

$$=\frac{\cancel{2x^2}-4x+6x-6-\cancel{2x^2}+2x}{-2\left(x^2-2x\right)^{\frac{3}{2}}}$$

$$=-\frac{(4x-6)}{2\left(x^2-2x\right)^{\frac{3}{2}}}$$

$$=-\frac{(2x-3)}{\left(x^2-2x\right)^{\frac{3}{2}}}$$

Check using the Product Rule.

$$y=(3-x)\left(x^2-2x\right)^{-\frac{1}{2}}$$

$$\begin{aligned}\frac{dy}{dx}&=(3-x)\frac{d}{dx}\left(x^2-2x\right)^{-\frac{1}{2}}+\left(x^2-2x\right)^{-\frac{1}{2}}\frac{d}{dx}(3-x)\\&=(3-x)\left(-\frac{1}{2}\right)\left(x^2-2x\right)^{-\frac{3}{2}}(2x-2)+\left(x^2-2x\right)^{-\frac{1}{2}}(-1)\\&=-\frac{1}{2}\left(x^2-2x\right)^{-\frac{3}{2}}(2x-2)(3-x)-\left(x^2-2x\right)^{-\frac{1}{2}}\\&=-\frac{1}{2}\left(x^2-2x\right)^{-\frac{3}{2}}\left[(2x-2)(3-x)+2\left(x^2-2x\right)\right]\\&=-\frac{1}{2}\left(x^2-2x\right)^{-\frac{3}{2}}\left(6x-\cancel{2x^2}-6+2x+\cancel{2x^2}-4x\right)\\&=-\frac{(4x-6)}{2\left(x^2-2x\right)^{\frac{3}{2}}}\\&=-\frac{(2x-3)}{\left(x^2-2x\right)^{\frac{3}{2}}}\end{aligned}$$

Before you do the following set of questions, watch the video entitled *The Product Rule and The Quotient Rule* from the *Catch 31* series, ACCESS Network. The second half of the program deals with the quotient rule. This video is available from Learning Resources Distributing Centre.

1. Find $\frac{dy}{dx}$ using the quotient rule for each of the following functions.

a. $y=\dfrac{3x-2}{5x+7}$

b. $y=\dfrac{2-3x+x^2}{5+x+2x^2}$

c. $y=\dfrac{5x-1}{3x-x^2}$

d. $y=\dfrac{x}{\sqrt{2x^2+1}}$

e. $y=\dfrac{2x-3}{\sqrt{x^2-3}}$

f. $y=\dfrac{-5x}{\sqrt{3-x^2}}$

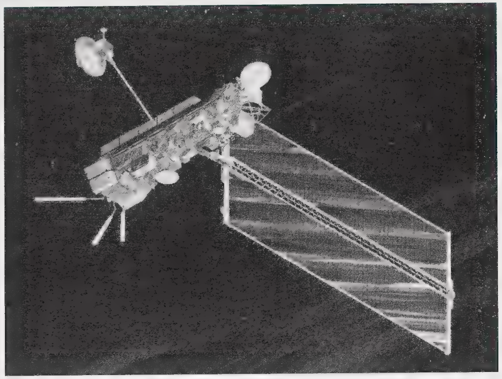
2. Find the derivative of $y=\frac{x^5}{x^2}$ by first using the quotient rule.

Compare your answer to what you would get if you were to divide first.

Check your answers by turning to the Appendix.

You will have to decide for yourself which you prefer: using the quotient rule, or changing the function to a product first and using the product rule.

Activity 8: Implicit Differentiation



NASA

What does differential calculus have to do with the stars?

The planets in Earth's solar system all travel in elliptical orbits about the sun; the orbits of communication satellites are almost circular; and the paths of some comets are parabolic or hyperbolic as they whip around the sun. Sir Isaac Newton's work in calculus and gravity helped answer questions about the orbits of the planets: the planets' speeds and locations. In this activity you will find derivatives of relations such as ellipses, circles, parabolas, and hyperbolas. These can be used to describe paths of the stars in the galaxies.

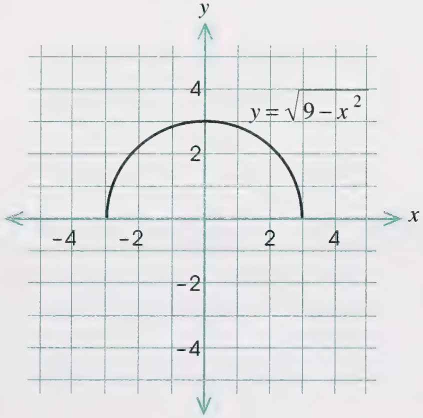
So far, the functions you have worked with have been of the form $y = f(x)$, equations which express y as an **explicit function** of x. You will now deal with equations such as $x^2 + y^2 = 5$, $x^2 - y^2 = 4$, $x^3 y^2 = 3$, and $xy = 8$. In these equations the relationship between x and y is not expressed explicitly, but rather, implicitly. These relationships contain one or more of what are termed **implicit functions**.

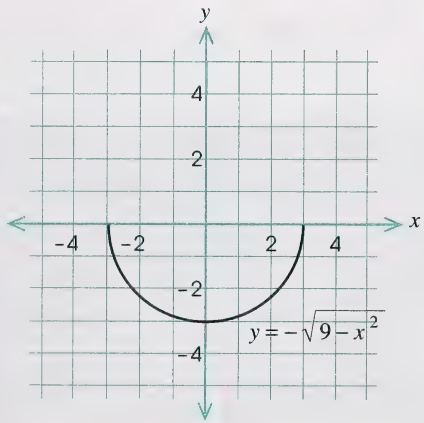
Occasionally, you will be able to take an implicit function and convert it into an explicit function of x. For instance, $xy = 8$ can be expressed as $y = \frac{8}{x}$. Other equations are more cumbersome to work with. Here is one sample.

$$x^2 + y^2 = 9$$
$$y^2 = 9 - x^2$$
$$y = \pm\sqrt{9 - x^2}$$

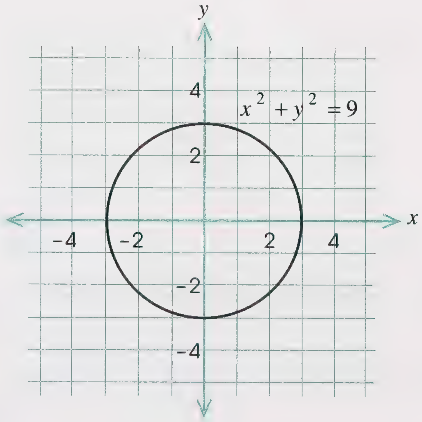
The original equation is a relation that detemines two explicit functions.

Each function defines a semicircle.





If you put the two semicircles together, they form the circle that is the graph of the original relation.



The original equation is called a **relation** rather than a function because y is not uniquely determined. You will recall the vertical-line test. If a vertical line can be drawn through no more than one point, the relation is called a function. If a vertical line can be drawn through more than one point, as it can for the circle, the relation is not a function. The relation may, in fact, define two or more explicit functions.

Not all equations which describe implicit relationships between x and y can be solved explicitly for y in terms of x. For example, it is not possible to rewrite equations such as $x^5-4xy-y^5=4$ in the form $y=f(x)$. Nevertheless, it is still possible to find the derivative $\frac{dy}{dx}$.

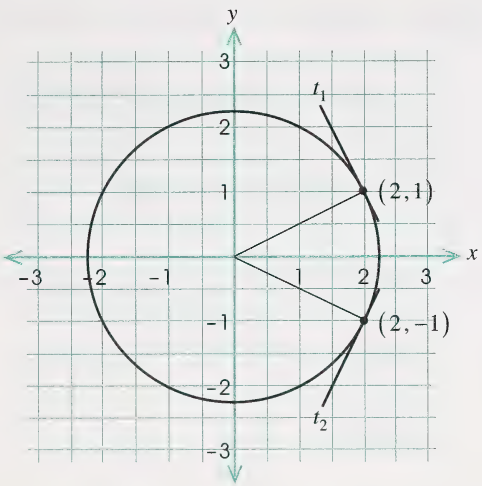
The process you will use to find the derivative of a relation is called **implicit differentiation**.

Example 1

Graph $x^2 + y^2 = 5$. Determine the slope of the tangent lines at $x = 2$ from the graph; by rewriting the relation, solving for y explictly in terms of x, and differentiating the result; and through implicit differentiation. Compare your results.

Solution

Method 1: Graphing



First, find the points of tangency.

Because $x^2 + y^2 = 5$, when $x = 2$,

$$\begin{aligned} 2^2 + y^2 &= 5 \\ y^2 &= 5 - 4 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

The points of tangency are $(2, 1)$ and $(2, -1)$.

The slope of the radius drawn to $(2, 1)$ is $\frac{\text{rise}}{\text{run}} = \frac{1}{2}$. Now, the tangent line, t_1, is perpendicular to the radius at $(2, 1)$. Therefore, the slope of t_1 is -2, the negative reciprocal of the slope of the radius.

The slope of the radius drawn to $(2, -1)$ is $\frac{\text{rise}}{\text{run}} = \frac{-1}{2}$. Now, the tangent line, t_2, is perpendicular to the radius at $(2, -1)$. Therefore, the slope of t_2 is 2, the negative reciprocal of the slope of the radius.

Method 2: Rewriting the Relation

Solve $x^2 + y^2 = 5$ for y in terms of x.

$$\begin{aligned} x^2 + y^2 &= 5 \\ y^2 &= 5 - x^2 \\ y &= \pm\sqrt{5 - x^2} \end{aligned}$$

$$\therefore\ y = \left(5 - x^2\right)^{\frac{1}{2}} \text{ and } y = -\left(5 - x^2\right)^{\frac{1}{2}}$$

$y = \left(5 - x^2\right)^{\frac{1}{2}}$ represents the upper semicircle.

To obtain the slope of the tangent t_1, differentiate and evaluate the derivative at $x = 2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}\left(5 - x^2\right)^{\frac{1}{2}-1}(-2x) \\ &= -x\left(5 - x^2\right)^{-\frac{1}{2}} \end{aligned} \qquad \begin{aligned} \therefore\ t_1 &= -2\left(5 - 2^2\right)^{-\frac{1}{2}} \\ &= -2(1) \\ &= -2 \end{aligned}$$

$y = -\left(5 - x^2\right)^{\frac{1}{2}}$ represents the lower semicircle.

To obtain the slope of the tangent t_2, differentiate and evaluate the derivative at $x = 2$.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{2}\left(5 - x^2\right)^{\frac{1}{2}-1}(-2x) \\ &= x\left(5 - x^2\right)^{-\frac{1}{2}} \end{aligned} \qquad \begin{aligned} \therefore\ t_2 &= 2\left(5 - 2^2\right)^{-\frac{1}{2}} \\ &= 2(1) \\ &= 2 \end{aligned}$$

This is the same result as on the graph.

Method 3: Implicit Differentiation

To differentiate implicitly, do **not** solve the equation for y in terms of x. Differentiate both sides of the relation with respect to x and then solve the derived relation for $\frac{dy}{dx}$.

$$\begin{aligned} x^2 + y^2 &= 5 \\ \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(5) \qquad \text{(chain rule)} \\ 2x + 2y\frac{dy}{dx} &= 0 \\ x + y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

Substitute the coordinates of the points of tangency into this derivative to find the slopes of the tangent lines.

At $(2,1)$, the slope of t_1 is $\frac{dy}{dx}=\frac{-2}{1}=-2$.

At $(2,-1)$, the slope of t_2 is $\frac{dy}{dx}=\frac{-2}{-1}=2$.

All three methods yield the same result!

Example 1 demonstrates the method of implicit differentiation. In this method you treat y as a differentiable function of x. Simply apply the rules for finding derivatives that you have already practised. Differentiate both sides of the relation; then solve for $\frac{dy}{dx}$.

Remember the following:

- Differentiate a term such as xy using the product rule.

$$\begin{aligned}\frac{d}{dx}(xy) &= x\frac{d}{dx}(y)+y\frac{d}{dx}(x)\\ &= x\frac{dy}{dx}+y\end{aligned}$$

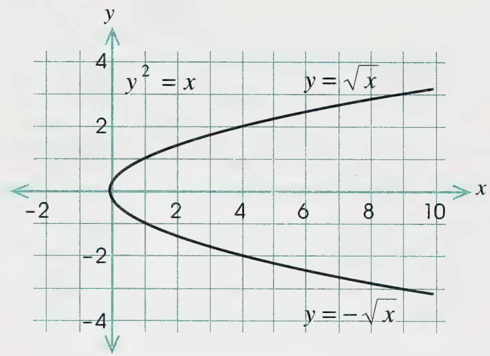
- Differentiate a power of y such as y^n using the chain rule.

$$\frac{d}{dx}\left(y^n\right)=ny^{n-1}\frac{dy}{dx}$$

Example 2

Differentiate $y^2=x$.

Solution



Method 1

Differentiate implicitly.

$$\begin{aligned}y^2 &= x\\ \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(x)\\ 2y\frac{dy}{dx} &= 1\\ \frac{dy}{dx} &= \frac{1}{2y}\end{aligned}$$

Method 2

Solve the relation explicitly for *y* in terms of *x*.

$$\begin{aligned} y^2 &= x \\ y &= \pm\sqrt{x} \\ y &= \pm x^{\frac{1}{2}} \\ \frac{dy}{dx} &= \pm\frac{1}{2}x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= \pm\frac{1}{2x^{\frac{1}{2}}} \end{aligned}$$

Are the two results equivalent?

By substitution show that $\frac{1}{2y} = \pm\frac{1}{2x^{\frac{1}{2}}}$.

LS	RS
$\frac{1}{2y}$	$\pm\frac{1}{2x^{\frac{1}{2}}}$
$= \frac{1}{2\left(\pm x^{\frac{1}{2}}\right)}$	
$= \pm\frac{1}{2x^{\frac{1}{2}}}$	

LS = RS

You should note that the plus and minus signs indicate that the original relation yields two explicit functions: $y = \sqrt{x}$ and $y = -\sqrt{x}$. Each of those functions has a derivative for a given positive value of *x*. In other words, $y^2 = x$ has two tangents for a given positive *x*-value.

Note: In the examples that follow, the symbol y' will be used instead of $\frac{dy}{dx}$.

Example 3

Determine $\frac{dy}{dx}$ for $2y^3 + y^2 - 2x^2 = 0$.

Solution

$$\begin{aligned} 2y^3 + y^2 - 2x^2 &= 0 \\ 6y^2y' + 2yy' - 4x &= 0 \\ 6y^2y' + 2yy' &= 4x \\ y'\left(6y^2 + 2y\right) &= 4x \\ y' &= \frac{4x}{6y^2 + 2y} \\ &= \frac{2x}{3y^2 + y} \end{aligned}$$

Example 4

Determine $\frac{dy}{dx}$ for $y^4 + y^3 = x^5 + x$.

Solution

In order to determine $\frac{dy}{dx}$ for $y^4 + y^3 = x^5 + x$, both sides of the equation are differentiated with respect to x.

$$\frac{d}{dx}\left(y^4 + y^3\right) = \frac{d}{dx}\left(x^5 + x\right)$$
$$4y^3\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 5x^4 + 1$$
$$\left(4y^3 + 3y^2\right)\frac{dy}{dx} = 5x^4 + 1$$
$$\frac{dy}{dx} = \frac{5x^4 + 1}{4y^3 + 3y^2}$$

Implicit differentiation can be used to obtain $\frac{dy}{dx}$ for the graphs of the quadratic relations.

Consider the following:

- the graph of a circle

$$x^2 + y^2 = 4$$

- the graph of an ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

- the graph of a hyperbola

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

- the graph of a parabola

$$y^2 = 16x$$

It should be pointed out that certain parabolas and certain hyperbolas are functions. In any case, take the derivative of the expression in order to find the slope.

Example 5

Find the derivative $\frac{dy}{dx}$ for the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which represents an ellipse.

Solution

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\left(\frac{dy}{dx}\right)=0$$

$$\frac{2y}{b^2}\left(\frac{dy}{dx}\right)=\frac{-2x}{a^2}$$

$$\frac{dy}{dx}=\frac{-2x}{a^2}\left(\frac{b^2}{2y}\right)$$

$$=\frac{-b^2x}{a^2y}$$

This is the value of the derivative. Note that implicit differentiation was used to find $\frac{dy}{dx}$.

If you want to find the slope at (x_1, y_1) for a tangent to an ellipse, you would write the following:

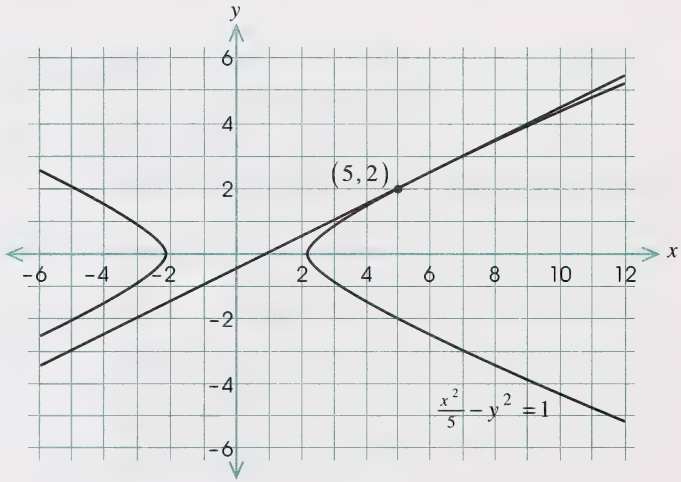
$$\frac{dy}{dx}=\frac{-b^2x_1}{a^2y_1}$$

Study the following example.

Example 6

Find the derivative $\frac{dy}{dx}$ of the relation $\frac{x^2}{5}-y^2=1$. What would be the slope of its tangent through $(5,2)$?

Solution



$$\frac{x^2}{5} - \frac{y^2}{1} = 1$$

$$\frac{2x}{5} - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x}{5}\left(\frac{1}{2y}\right)$$

$$= \frac{x}{5y}$$

The derivative of the relation is $\frac{x}{5y}$.

The slope of a tangent line at point $(5, 2)$ is as follows:

$$\frac{dy}{dx} = \frac{x}{5y}$$

$$= \frac{5}{5(2)}$$

$$= \frac{1}{2}$$

The slope of the tangent would be $\frac{1}{2}$.

Example 7

Find the derivative of y with respect to x for the relation $2x^2 + xy - y^2 = -7$. Find the value of the derivative $\frac{dy}{dx}$ at $(-2, 3)$ on the curve. State where $\frac{dy}{dx}$ does not exist.

Solution

$$2x^2 + xy - y^2 = -7$$

$$4x + y(1) + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = -4x - y$$

$$(x - 2y)\frac{dy}{dx} = -(4x + y)$$

$$\frac{dy}{dx} = \frac{-(4x + y)}{x - 2y}$$

The derivative of the relation is $\frac{-(4x+y)}{x-2y}$.

The derivative at $(-2, 3)$ is as follows:

$$\frac{dy}{dx} = \frac{-(4x + y)}{x - 2y}$$

$$= \frac{-[4(-2) + 3]}{(-2) - 2(3)}$$

$$= \frac{-8 + 3}{2 + 6}$$

$$= \frac{-5}{8}$$

The derivative is not defined where $x - 2y = 0$. That is, when $x = 2y$.

Now that you know how to find the derivative to a relation, proceed to find the equation of the tangent to a relation.

Example 8

Find the equation of the tangent line to the relation $\frac{x^2}{4} + \frac{y^2}{9} = 2$.

The tangent line passes through the point $P(-2, 3)$.

Solution

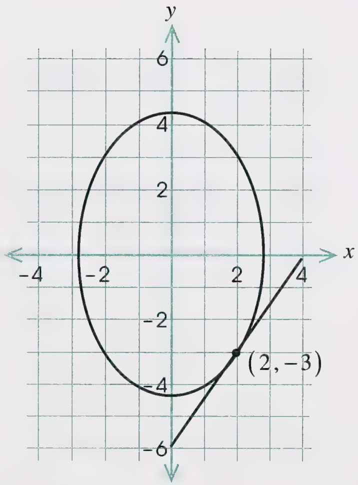
$$\frac{x^2}{4} + \frac{y^2}{9} = 2$$

$$\frac{2x}{4} + \frac{2y}{9}\left(\frac{dy}{dx}\right) = 0$$

$$\frac{2y}{9}\left(\frac{dy}{dx}\right) = -\frac{x}{2}$$

$$\frac{dy}{dx} = \frac{-x}{2}\left(\frac{9}{2y}\right)$$

$$= \frac{-9x}{4y}$$



The slope of the line at $P(2, -3)$ is as follows:

$$\frac{dy}{dx} = \frac{-9(2)}{4(-3)}$$

$$= \frac{-18}{-12}$$

$$= \frac{18}{12}$$

$$= \frac{3}{2}$$

In order to find the equation of the tangent line, substitute $m = \frac{3}{2}$ and $P(2, -3)$ in the slope-point formula.

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = \frac{3}{2}(x - 2)$$

$$y = \frac{3}{2}x - 6$$

$$3x - 2y - 12 = 0$$

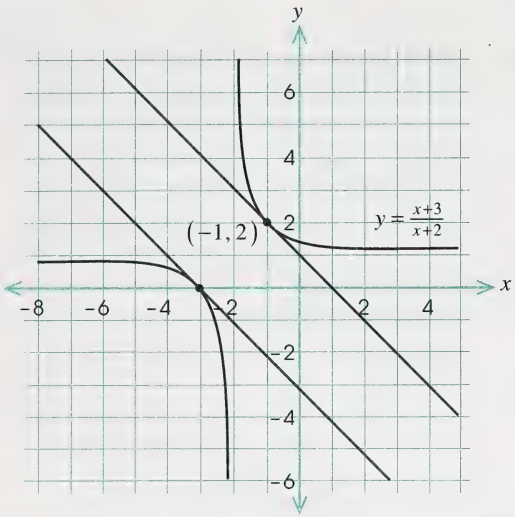
Therefore, the equation of the tangent line is $3x - 2y - 12 = 0$.

The questions involving relations and tangents are not always this easy. You should try one that is in the form of a quotient.

Example 9

Find the equation of the tangent to the graph of $y = \frac{x+3}{x+2}$ at the point $(-1, 2)$. Find the equation for another tangent if it is parallel to the first tangent.

Solution



Note that $y = \frac{x+3}{x+2}$ becomes $xy + 2y = x + 3$.

$$y(1) + x\frac{dy}{dx} + 2\frac{dy}{dx} = 1$$
$$(x+2)\frac{dy}{dx} = 1 - y$$
$$\frac{dy}{dx} = \frac{1-y}{x+2}$$

The slope at $(-1, 2)$ is as follows:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1-y}{x+2}\\ &= \frac{1-2}{-1+2}\\ &= -1\end{aligned}$$

Use the slope-point formula to find the equation of the tangent line.

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - 2 &= -1[x - (-1)]\\ y - 2 &= -x - 1\\ x + y - 1 &= 0\end{aligned}$$

The equation of the tangent at $(-1, 2)$ is $x + y - 1 = 0$.

For $\frac{dy}{dx} = -1$ other values of x and y may exist. If there is a tangent parallel to the first line, then its slope is -1.

Therefore, $-1 = \frac{1-y}{x+2}$ where $y = \frac{x+3}{x+2}$.

$$-1 = \frac{1 - \left(\frac{x+3}{x+2}\right)}{x+2}$$

$$-1 = \frac{x+2-(x+3)}{(x+2)^2}$$

$$\frac{x+2-x-3}{(x+2)^2} = -1$$

$$\frac{-1}{(x+2)^2} = -1$$

$$(x+2)^2 = 1$$

$$x+2 = \pm 1$$

$\therefore\ x = -3$ or $x = -1$

For $x = -1$, $y = 2$. Therefore, the first point of tangency given is $Q(-1, 2)$.

For $x = -3$, $y = \frac{-3+3}{-3+2} = \frac{0}{-1} = 0$.

The second point of tangency is $(-3, 0)$. The equation of the tangent at $(-3, 0)$ is $y = -x - 3$.

Before doing the following questions, watch the video entitled *Derivatives for Relations or Functions Defined Implicitly* from the *Catch 31* series, ACCESS Network. This video gives further historical background, investigates other applications of implicit differentiation, and reviews algebraic procedures. Make notes on the proof of the power rule for rational exponents using implicit differentiation. This video is available from Learning Resources Distributing Centre.

1. Find y' for the following relations.

a. $3xy + 2x^2 + y^2 = 2$

b. $y^3 - 3xy^2 - 2x^2y + x = 2$

c. $\frac{x^{\frac{1}{3}}}{a} - \frac{y^{\frac{1}{3}}}{a} = 1$

d. $xy^2 = 2x - \frac{1}{x}$

2. Write the equation of the tangent to the graph of the relation $x^2 + 4xy - 2y = 3$ at the point $P(1,1)$.

3. Find the tangents to the relation $x^2 + y^2 = 169$ (a circle) at the points $P(5,12)$ and $Q(5,-12)$. At what point *T* do they intersect?

Check your answers by turning to the Appendix.

Activity 9: Derivatives of Higher Order

Part of the excitement of downhill skiing is the exhilaration of speed and acceleration. As a skier moves downhill, the position changes. The position s of the skier can be described by a function of time t.

$$s = f(t)$$

You will recall that the change in position, or velocity v, at any time t is the derivative of the position function.

$$v = f'(t) \text{ or } \frac{ds}{dt}$$

But the velocity changes as well! The rate of change in velocity, or acceleration a, must be a derivative as well—the derivative of velocity.

$$a = \frac{dv}{dt} \text{ or } \frac{d}{dt}\left(f'(t)\right)$$

The intricacies of position, velocity, and acceleration will be discussed in Module 6. This activity deals with the process of finding the derivative of a derivative, of which acceleration is an example.

If $y = f(x)$ is a differentiable function of x, then the derivative $\frac{dy}{dx} = f'(x)$ is a function of x. You should be able to find the derivative of this function as well.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(f'(x)\right)$$

This **second derivative** is represented in Leibniz notation as $\frac{d^2x}{dx^2}$. You should not confuse this notational device with exponents. The symbol simply means that the derivative has been taken a second time. Other notations are y'', $f''(x)$, $D^2 f(x)$, and $D_x^2 y$.

If the derivative is taken a third time, use any one of $\frac{d^3y}{dx^3}$, y''', $f'''(x)$, $D^3 f(x)$, or $D_x^3 y$.

For fourth derivatives and higher, the primes (′) are not used; rather, the number of the derivative appears in parentheses as a superscript:

$$y^{(4)} \text{ and } f^{(4)}(x), \text{ and so on}$$

Second, third, fourth (etc.) derivatives are all called higher-order derivatives. The following examples illustrate techniques for finding those derivatives.

Example 1

If $y = x^3 - 2x^2$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution

$$\frac{dy}{dx} = 3x^{3-1} - 2\left(2x^{2-1}\right)$$

Differentiate term by term.

$$= 3x^2 - 4x$$

$$\frac{d^2y}{dx^2} = 3\left(2x^{2-1}\right) - 4(1)$$

Differentiate a second time.

$$= 6x - 4$$

Example 2

Find y''' if $y = \frac{x+1}{x-1}$.

Solution

Use the quotient rule.

$$y' = \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(1+0) - (x+1)(1-0)}{(x-1)^2}$$

$$= \frac{x-1-x-1}{(x-1)^2}$$

$$= -2(x-1)^{-2}$$

$$y'' = -2(-2)(x-1)^{-2-1}(1)$$

$$= 4(x-1)^{-3}$$

$$y''' = 4(-3)(x-1)^{-3-1}$$

$$= -12(x-1)^{-4}$$

1. Determine $f'''(2)$ if $f(x) = 5x^3 - 3x$.
2. Given $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, find $f^{(n+1)}(x)$.
3. Find y'' if $y = (x-1)\sqrt{2x+1}$.
4. Show that $y^{(5)} = 5!$, if $y = x^5$.

Check your answers by turning to the Appendix.

Example 3

Find y'' if $y^2 - xy = 3$.

Solution

Differentiate implictly.

$$\begin{aligned} 2yy' - \left(x\frac{d}{dx}(y) + y\frac{d}{dx}(x) \right) &= 0 \\ 2yy' - (xy' + y) &= 0 \\ 2yy' - xy' - y &= 0 \\ (2y - x)y' &= y \\ y' &= \frac{y}{2y - x} \end{aligned}$$

Apply the quotient rule.

$$\begin{aligned} y'' &= \frac{(2y-x)\frac{d}{dx}(y) - y\frac{d}{dx}(2y-x)}{(2y-x)^2} \\ &= \frac{(2y-x)y' - y(2y'-1)}{(2y-x)^2} \\ &= \frac{2yy' - xy' - 2yy' + y}{(2y-x)^2} \\ &= \frac{y - xy'}{(2y-x)^2} \end{aligned}$$

Replace y' by $\frac{y}{2y-x}$.

$$\begin{aligned} y'' &= \frac{y - x\left(\frac{y}{2y-x}\right)}{(2y-x)^2} \\ &= \frac{\frac{y(2y-x)-xy}{2y-x}}{(2y-x)^2} \\ &= \frac{2y^2 - xy - xy}{(2y-x)^3} \\ &= \frac{2y^2 - 2xy}{(2y-x)^3} \end{aligned}$$

Example 4

Find y'' at $(1, 2)$, given $5xy + y^2 = x^3$.

Solution

Differentiate implicitly.

$$\begin{aligned} 5x\frac{d}{dx}(y) + y\frac{d}{dx}(5x) + 2y\frac{dy}{dx} &= 3x^2 \\ 5xy' + 5y + 2yy' &= 3x^2 \end{aligned}$$

$$5xy' + 2yy' = 3x^2 - 5y$$
$$(5x + 2y)y' = 3x^2 - 5y$$
$$y' = \frac{3x^2 - 5y}{5x + 2y}$$

Apply the quotient rule.

$$y'' = \frac{(5x+2y)\frac{d}{dx}\left(3x^2 - 5y\right) - \left(3x^2 - 5y\right)\frac{d}{dx}(5x+2y)}{(5x+2y)^2}$$
$$= \frac{(5x+2y)(6x - 5y') - \left(3x^2 - 5y\right)(5 + 2y')}{(5x+2y)^2}$$

This could be simplified; however, to find y'' at $(1, 2)$, simply replace x by 1, y by 2, and y' by

$$\frac{3x^2 - 5y}{5x + 2y} = \frac{3(1)^2 - 5(2)}{5(1) + 2(2)}$$
$$= \frac{3 - 10}{5 + 4}$$
$$= \frac{-7}{9}$$

At $(1, 2)$,

$$y'' = \frac{[5(1) + 2(2)]\left[6(1) - 5\left(\frac{-7}{9}\right)\right] - \left[3(1)^2 - 5(2)\right]\left[5 + 2\left(\frac{-7}{9}\right)\right]}{[5(1) + 2(2)]^2}$$
$$= \frac{[5+4]\left[6 + \frac{35}{9}\right] - [3 - 10]\left[5 - \frac{14}{9}\right]}{9^2}$$
$$= \frac{9\left[\frac{54}{9} + \frac{35}{9}\right] + 7\left[\frac{45}{9} - \frac{14}{9}\right]}{81}$$
$$= \frac{9\left[\frac{89}{9}\right] + 7\left[\frac{31}{9}\right]}{81}$$
$$= \frac{1018}{729}$$

5. Find $\frac{d^2y}{dx^2}$ if $y^2 + y = x$.

6. Find y'' if $y^2 = x^3$.

Check your answers by turning to the Appendix.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

When differentiating, express your answers, whenever possible, in factored form. This form makes it easier for you to determine critical points on the graph of a function.

Difficulties may arise when removing common factors with either negative or fractional exponents. The following examples should clarify the steps in removing factors of this type. Remember, all of these examples are simply extensions of the following rule:

$$ab + ac = a(b+c)$$

Example 1

Factor $(x+3)^4(x+1)+(x+3)^3(x+1)^2$.

Solution

Compare $(x+3)^4$ with $(x+3)^3$. Which of the two powers has the smaller exponent? $(x+3)^3$ is part of the highest common factor.

Compare $(x+1)$ with $(x+1)^2$. Which of these powers has the smaller exponent? $(x+1)$ is part of the common factor.

Remove $(x+3)^3(x+1)$ as the highest common factor.

$$\begin{aligned}&(x+3)^4(x+1)+(x+3)^3(x+1)^2\\ &\quad=(x+3)^3(x+1)[(x+3)+(x+1)]\end{aligned}$$

Square brackets are used for clarity; they separate the common factor from the sum or difference it multiplies. Remove the parentheses within the square brackets, and combine similar terms.

$$\begin{aligned}&=(x+3)^3(x+1)[x+3+x+1]\\ &=(x+3)^3(x+1)[2x+4]\end{aligned}$$

Remove 2 as a common factor from $2x+4$. Monomial common factors are normally placed as the leading factor.

$$=2(x+3)^3(x+1)(x+2)$$

Example 2

Factor $(x-3)^{-2}(x-4)^{-4}-(x-3)^{-3}(x-4)^{-3}$.

Solution

Compare powers. The power with the smaller exponent is removed as part of the common factor. Compare $(x-3)^{-2}$ and $(x-3)^{-3}$; because -3 is less than -2, $(x-3)^{-3}$ is removed as a common factor. Similarly, when $(x-4)^{-3}$ and $(x-4)^{-4}$ are compared, because -4 is less than -3, $(x-4)^{-4}$ is removed.

$-4+1=-3$

$$
\begin{aligned}
&(x-3)^{-2}(x-4)^{-4}-(x-3)^{-3}(x-4)^{-3}\\
&\quad=(x-3)^{-3}(x-4)^{-4}\left[(x-3)^{1}-(x-4)^{1}\right]
\end{aligned}
$$

$-3+1=-2$

$$
\begin{aligned}
&=(x-3)^{-3}(x-4)^{-4}[x-3-x+4]\\
&=(x-3)^{-3}(x-4)^{-4}[1]\\
&=(x-3)^{-3}(x-4)^{-4}
\end{aligned}
$$

Example 3

Factor $(2x-1)^{-\frac{1}{2}}(3x+1)^{\frac{1}{3}}-(2x-1)^{\frac{1}{2}}(3x+1)^{-\frac{2}{3}}$.

Solution

As in the first two examples, look for the smaller exponent.

$$
\begin{aligned}
&(2x-1)^{-\frac{1}{2}}(3x+1)^{\frac{1}{3}}-(2x-1)^{\frac{1}{2}}(3x+1)^{-\frac{2}{3}}\\
&\quad=(2x-1)^{-\frac{1}{2}}(3x+1)^{-\frac{2}{3}}\left[(3x+1)^{1}-(2x-1)^{1}\right]
\end{aligned}
$$

It is always a good idea to check your work by multiplying back at this point. You should get the original expression!

$$
\begin{aligned}
&=(2x-1)^{-\frac{1}{2}}(3x+1)^{-\frac{2}{3}}[3x+1-2x+1]\\
&=(2x-1)^{-\frac{1}{2}}(3x+1)^{-\frac{2}{3}}(x+2)\\
&=(2x-1)^{-\frac{1}{2}}(3x+1)^{-\frac{2}{3}}(x+2)
\end{aligned}
$$

Example 4

Differentiate $y=(x+1)^{3}(x-4)^{-2}(2x-1)^{\frac{1}{2}}$ using the product rule. Express your answer in factored form.

Solution

Because the function consists of three factors, two factors are grouped as a single factor initially. It does not matter which two are chosen.

If $(x+1)^3$ and $(x-4)^{-2}$ are grouped together,

$$\begin{aligned}
\frac{dy}{dx} &= (x+1)^3(x-4)^{-2}\frac{d}{dx}(2x-1)^{\frac{1}{2}} + (2x-1)^{\frac{1}{2}}\frac{d}{dx}\left[(x+1)^3(x-4)^{-2}\right] \\
&= (x+1)^3(x-4)^{-2}\left(\frac{1}{\cancel{2}}\right)(2x-1)^{-\frac{1}{2}}(\cancel{2}) + (2x-1)^{\frac{1}{2}}\left[(x+1)^3\frac{d}{dx}(x-4)^{-2} + (x-4)^{-2}\frac{d}{dx}(x+1)^3\right] \\
&= (x+1)^3(x-4)^{-2}(2x-1)^{-\frac{1}{2}} + (2x-1)^{\frac{1}{2}}\left[(x+1)^3(-2)(x-4)^{-3} + (x+4)^{-2}(3)(x+1)^2\right] \\
&= (x+1)^3(x-4)^{-2}(2x-1)^{-\frac{1}{2}} - 2(x+1)^3(x-4)^{-3}(2x-1)^{\frac{1}{2}} + 3(x+1)^2(x-4)^{-2}(2x-1)^{\frac{1}{2}}
\end{aligned}$$

Ready to factor?

$$\begin{aligned}
&= (x-4)^{-3}(x+1)^2(2x-1)^{-\frac{1}{2}}\left[(x-4)(x+1) - 2(x+1)(2x-1) + 3(x-4)(2x-1)\right] \\
&= (x-4)^{-3}(x+1)^2(2x-1)^{-\frac{1}{2}}\left[\left(x^2-3x-4\right) - 2\left(2x^2+x-1\right) + 3\left(2x^2-9x+4\right)\right] \\
&= (x-4)^{-3}(x+1)^2(2x-1)^{-\frac{1}{2}}\left[x^2-3x-4-4x^2-2x+2+6x^2-27x+12\right] \\
&= (x-4)^{-3}(x+1)^2(2x-1)^{-\frac{1}{2}}\left[3x^2-32x+10\right]
\end{aligned}$$

Factor the following:

1. **a.** $x^{-3}(2x-1) + x^{-4}(2x-1)^2$

   b. $x^{\frac{1}{2}}(x+1)^{-2} + x^{\frac{3}{2}}(x+1)^{-1}$

   c. $\left(\sqrt{x+1}\right)(2x-3)^{-1} + \frac{1}{\sqrt{x+1}}(2x-3)^{-2}$

2. Differentiate $y = x^2 (x-3)^{-1} (x+1)^3$ and express your answer in factored form.

Check your answers by turning to the Appendix.

Enrichment

In this section you have used the product rule to find derivatives. You will recall that if $y = uv$, where u and v are differentiable functions of x, then $\frac{dy}{dx} = u\frac{dv}{dy} + v\frac{du}{dx}$.

Can you extend this rule to find higher-order derivatives?

To find $\frac{d^2 y}{dx^2}$, apply the product rule to each term of the first derivative.

product

$$\begin{aligned}
\frac{d^2 y}{dx^2} &= \frac{d}{dx}\left(u \bullet \frac{dv}{dx}\right) + \frac{d}{dx}\left(v \bullet \frac{du}{dx}\right) \\
&= \left[u\frac{d}{dx}\left(\frac{dv}{dx}\right) + \frac{du}{dx} \bullet \frac{dv}{dx}\right] + \left[\frac{dv}{dx} \bullet \frac{du}{dx} + v\frac{d}{dx}\left(\frac{du}{dx}\right)\right] \\
&= u\frac{d^2 v}{dx^2} + \frac{du}{dx} \bullet \frac{dv}{dx} + \frac{du}{dx} \bullet \frac{dv}{dx} + v\frac{d^2 u}{dx^2} \\
&= u\frac{d^2 v}{dx^2} + 2 \bullet \frac{du}{dx} \bullet \frac{dv}{dx} + v\frac{d^2 u}{dx^2}
\end{aligned}$$

Now find $\frac{d^3y}{dx^3}$. Again, apply the product rule to each term.

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(u\frac{d^2v}{dx^2}\right) + \frac{d}{dx}\left(2 \bullet \frac{du}{dx} \bullet \frac{dv}{dx}\right) + \frac{d}{dx}\left(v\frac{d^2u}{dx^2}\right)$$

$$= u\frac{d}{dx}\left(\frac{d^2v}{dx^2}\right) + \frac{du}{dx} \bullet \frac{d^2v}{dx^2} + 2\left[\frac{du}{dx} \bullet \frac{d}{dx}\left(\frac{dv}{dx}\right) + \frac{dv}{dx} \bullet \frac{d}{dx}\left(\frac{du}{dx}\right)\right] + \frac{dv}{dx} \bullet \frac{d^2u}{dx^2} + v\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)$$

$$= u\frac{d^3v}{dx^3} + \frac{du}{dx} \bullet \frac{d^2v}{dx^2} + 2 \bullet \frac{du}{dx} \bullet \frac{d^2v}{dx^2} + 2 \bullet \frac{dv}{dx} \bullet \frac{d^2u}{dx} + \frac{dv}{dx} \bullet \frac{d^2u}{dx^2} + v\frac{d^3u}{dx^3}$$

$$= u\frac{d^3v}{dx^3} + 3 \bullet \frac{du}{dx} \bullet \frac{d^2v}{dx^2} + 3 \bullet \frac{dv}{dx} \bullet \frac{d^2u}{dx} + v\frac{d^3u}{dx^3}$$

If you summarize the results so far and include the rule for $\frac{d^4y}{dx^4}$, you should recognize a familiar pattern.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2}{dx^2}(uv) = u\frac{d^2v}{dx^2} + 2 \bullet \frac{du}{dx} \bullet \frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

$$\frac{d^3}{dx^3}(uv) = u\frac{d^3v}{dx^3} + 3 \bullet \frac{du}{dx} \bullet \frac{d^2v}{dx^2} + 3 \bullet \frac{d^2u}{dx^2} \bullet \frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

$$\frac{d^4}{dx^4}(uv) = u\frac{d^4v}{dx^4} + 4 \bullet \frac{du}{dx} \bullet \frac{d^3v}{dx^3} + 6 \bullet \frac{d^2u}{dx^2} \bullet \frac{d^2v}{dx^2} + 4 \bullet \frac{d^3u}{dx^3} \bullet \frac{dv}{dx} + v\frac{d^4u}{dx^3}$$

Before going any further, do question 1.

1. Derive the result for $\frac{d^4}{dx^4}(uv)$ by differentiating

$$\frac{d^3}{dx^3}(uv) = u\frac{d^3v}{dx^3} + 3\bullet\frac{du}{dx}\bullet\frac{d^2v}{dx^2} + 3\bullet\frac{d^2u}{dx^2}\bullet\frac{dv}{dx} + v\frac{d^3u}{dx^3}.$$

Check your answer by turning to the Appendix.

Notice that the results are identical in format to the Binomial Theorem.

Binomial Theorem	Product Rule
$(a+b)^1 = a^1 + b^1$	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$(a+b)^2 = a^2 + 2ab + b^2$	$\frac{d^2}{dx^2}(uv) = u\frac{d^2v}{dx^2} + 2\bullet\frac{du}{dx}\bullet\frac{dv}{dx} + v\frac{d^2u}{dx^2}$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$\frac{d^3}{dx^3}(uv) = u\frac{d^3v}{dx^3} + 3\bullet\frac{du}{dx}\bullet\frac{d^2v}{dx^2} + 3\bullet\frac{d^2u}{dx^2}\bullet\frac{dv}{dx} + v\frac{d^3u}{dx^3}$

2. Compare $(a+b)^4$ with $\frac{d^4}{dx^2}(uv)$.

Check your answer by turning to the Appendix.

You will recall that the coefficients of the binomial expansion can be obtained from Pascal's triangle.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Each entry in each row can be found by adding together the two entries from the row above it and that are on either side of the required value.

3. What would be the coefficients for $(a+b)^5$?

Check your answer by turning to the Appendix.

The coefficients in Pascal's triangle may also be written as combinations.

$$\binom{0}{0} \quad \text{where } \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$\binom{1}{0} \binom{1}{0}$$

$$\binom{2}{0} \binom{2}{1} \binom{2}{2}$$

$$\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$$

$$\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}$$

The general statement for the Binomial Theorem is as follows:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{n}b^n,$$

where the k^{th} term in the expansion is $\binom{n}{k-1}a^{n-k+1}b^{k-1}$.

For the n^{th}-order derivative of $y = uv$, the expansion is as follows:

$$\frac{d^n y}{dx^n} = \binom{n}{0}u\frac{d^n v}{dx^n} + \binom{n}{1}\frac{d^1 u}{dx^1}\bullet\frac{d^{n-1}v}{dx^{n-1}} + \binom{n}{2}\frac{d^2 u}{dx^2}\bullet\frac{d^{n-2}v}{dx^{n-2}} + \binom{n}{3}\frac{d^3 u}{dx^3}\bullet\frac{d^{n-3}v}{dx^{n-3}} + \ldots + \binom{n}{n}v\frac{d^n u}{dx^n},$$

where the k^{th} term of the derivative is $\binom{n}{k-1}\frac{d^{k-1}u}{dx^{k-1}}\bullet\frac{d^{n-k+1}v}{dx}$,

provided you agree to call $\frac{d^0 u}{dx^0} = u$ and $\frac{d^0 v}{dx^0} = v$. This rule is called **Leibniz's rule**.

Example 1

Use Leibniz's rule to find $\frac{d^2 y}{dx^2}$ if $y = x^3 \bullet (x-1)^2$.

Solution

Let $u = x^3$. Let $v = (x-1)^2$.

$\therefore \dfrac{du}{dx} = 3x^2$ $\qquad \dfrac{dv}{dx} = 2(x-1) = 2x-2$

$\dfrac{d^2u}{dx^2} = 6x$ $\qquad \dfrac{d^2v}{dx^2} = 2$

Substitute the preceding values into the following:

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2 \bullet \frac{du}{dx} \bullet \frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= x^3 \bullet 2 + 2 \bullet 3x^2 \bullet (2x-2) + (x-1)^2 \bullet 6x \\ &= 2x^3 + 6x^2(2x-2) + 6x\left(x^2 - 2x + 1\right) \\ &= 2x^3 + 12x^3 - 12x^2 + 6x^3 - 12x^2 + 6x \\ &= 20x^3 - 24x^2 + 6x\end{aligned}$$

This result may be verified as follows:

$$\begin{aligned}y &= x^3 \bullet (x-1)^2 \\ &= x^3\left(x^2 - 2x + 1\right) \\ &= x^5 - 2x^4 + x^3\end{aligned}$$

$$\frac{dy}{dx} = 5x^4 - 8x^3 + 3x^2$$

$$\frac{d^2y}{dx^2} = 20x^3 - 24x^2 + 6x$$

Example 2

Use Leibniz's rule to find $\frac{d^3y}{dx^3}$ if $y = \sqrt{2x-1} \bullet x$.

Solution

Let $u = (2x-1)^{\frac{1}{2}}$.

$$\begin{aligned}\frac{du}{dx} &= \frac{1}{2}(2x-1)^{-\frac{1}{2}}(2) \\ &= (2x-1)^{-\frac{1}{2}}\end{aligned} \qquad \begin{aligned}\frac{d^2u}{dx^2} &= -\frac{1}{2}(2x-1)^{-\frac{3}{2}}(2) \\ &= -(2x-1)^{-\frac{3}{2}}\end{aligned}$$

$$\begin{aligned}\frac{d^3u}{dx^3} &= \frac{3}{2}(2x-1)^{-\frac{5}{2}}(2) \\ &= 3(2x-1)^{-\frac{5}{2}}\end{aligned}$$

Let $v = x$.

$\therefore \dfrac{dv}{dx} = 1$ $\qquad \dfrac{d^2v}{dx^2} = 0$ $\qquad \dfrac{d^3v}{dx^3} = 0$

Substitute the preceding values into the following:

$$\frac{d^3y}{dx^3} = u\frac{d^3v}{dx^2} + 3\bullet\frac{du}{dx}\bullet\frac{d^2v}{dx^2} + 3\bullet\frac{d^2u}{dx^2}\bullet\frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

$$= (2x-1)^{\frac{1}{2}}(0) + 3(2x-1)^{-\frac{1}{2}}(0) + 3\left(-(2x-1)^{-\frac{3}{2}}\right)(1) + x\left(3(2x-1)^{-\frac{5}{2}}\right)$$

$$= -3(2x-1)^{-\frac{3}{2}} + 3x(2x-1)^{-\frac{5}{2}}$$

$$= 3(2x-1)^{-\frac{5}{2}}\left[-(2x-1)+x\right]$$

$$= 3(2x-1)^{-\frac{5}{2}}(-2x+1+x)$$

$$= 3(2x-1)^{-\frac{5}{2}}(1-x)$$

Now try the following questions.

4. Given $y = x^3\sqrt{1+2x}$, find $\frac{d^2y}{dx^2}$.

5. Determine $\frac{d^3y}{dx^3}$ if $y = \frac{x}{x+1}$.

Check your answers by turning to the Appendix.

Conclusion

In this section, you defined the derivative of a function, determined the circumstances for which the derivative does or does not exist, developed techniques for differentiating algebraic functions, and interpreted the derivative from the contexts of slope, rate, and function.

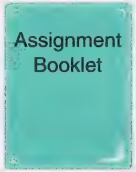
Check the following list of skills you should have mastered.

You should be able to do the following:

- find the derivative from first principles
- use $f'(x)$, y', and $\frac{dy}{dx}$ as alternate notations for the first derivative
- apply the power, sum, and difference, product, quotient, and chain rules, singly or in combination
- use the technique of implicit differentiation
- determine higher-order derivatives

The notation $\frac{dy}{dx}$ that you used in this section was developed by Leibniz in the seventeenth century. Since its introduction, this notation was used throughout continental Europe. In England, however, the notation that Newton used, $\dot{y}$, was used until the nineteenth century. Leibniz notation was finally accepted in England when it was feared a failure to change would hinder progress in mathematics and the sciences.

Assignment



You are now ready to complete the section assignment.

Module Summary

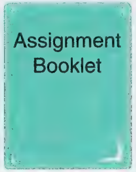
In this module you have begun your study of differential calculus. In Section 1, the derivative was defined and interpreted graphically. A tangent was defined as the limiting position for a sequence of secant lines. In turn, the derivative was defined using tangents. Normal lines were introduced; their slopes and equations were found using derivatives and tangents.

In Section 2 you derived, interpreted, and applied the rules for derivatives.

The derivative is used to model change. And change is universal. Films of life along a coral reef, for example, show schools of exotic fish swimming in and out of view, seeming to alter direction in unison. Each frame of the film represents a slice of time and motion, never to be repeated in precisely the same way.

Leibniz and Newton pioneered calculus to help us describe that universe in flux. In your future work you will use the techniques and symbolisms of this module as part of a shared language that science uses to speak about the world.

Final Module Assignment



You are now ready to complete the final module assignment.

APPENDIX

Glossary

Suggested Answers

Glossary

Chain rule: a method for finding the derivative of a composite function

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

Derivative: the function which yields the slope of a curve at any point; the instantaneous rate of change of a function with respect to the independent variable

Difference rule: the process of finding the derivative of the difference of two functions

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

Differentiable: capable of being differentiated at x if x is a member of the domain of the derivative of the function

Differentiation: the process of finding a derivative

Explicit function: a function expressed in the form $y = f(x)$

First principles: the process of finding the derivative from the definition

Higher-order derivative: any derivative of another derivative

Implicit differentiation: the process of finding the derivative of a relation, when the relationship between the variables is stated implicitly

Implicit function: a function in which the relationship between x and y is not stated explicitly as $y = f(x)$, but rather is implied

Normal: a line perpendicular to a curve at a given point

Power rule: the process of finding the derivative of a power, $y = x^n$

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Product rule: the process of finding the derivative of the product of two functions

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient rule: the process of finding the derivative of the quotient of two functions

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Secant: a line that intersects a curve at two or more points

Stationary point: a point on the graph of a function where the slope is zero

Sum rule: the process of finding the derivative of the sum of two functions

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$

Tangent: a line that touches a curve at a single point

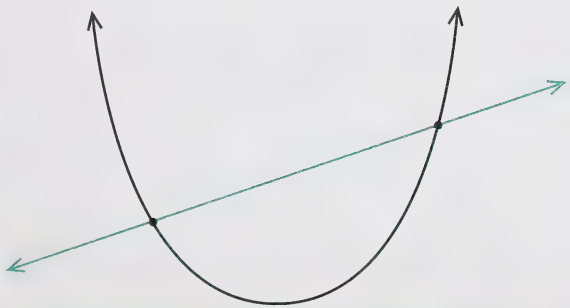
The tangent at a point is the limiting position of a sequence of secant lines that contains that point.

Suggested Answers

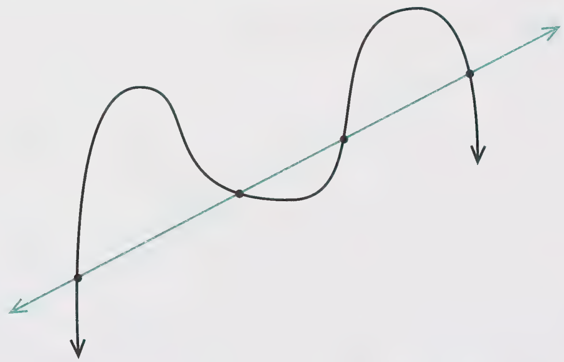
Section 1: Activity 1

1. Answers will vary. Only one curve is given as an example for question a. and one for question b.

a.



b.



2. **a.** You cannot draw a secant line because by definition, the secant line must intersect a curve in at least two places.

   b. AB is a tangent. It only touches the curve; a secant must intersect the curve in two or more places. PQ intersects the curve, but only at one point. It must intersect the curve twice before it can be considered a secant.

3. $(x_1, y_1)=(2,3)$ and $(x_2, y_2)=(4,6)$

$$m=\frac{y_2-y_1}{x_2-x_1}$$
$$=\frac{6-3}{4-2}$$
$$=\frac{3}{2}$$

y increases by 3 when x increases by 2; or, y increases by $\frac{3}{2}$ when x increases by 1.

4. Compare $y=-1x+3$ with the slope-y-intercept formula $y=mx+b$. $m=-1$. Therefore, y decreases by 1 when x increases by 1.

5. $$m=\frac{y_2-y_1}{x_2-x_1}=\frac{7-(-2)}{2-1}$$
$$=\frac{9}{1}$$
$$=9$$

The slope of the secant AB is 9.

6. Since you need to use the formula $m=\frac{y_2-y_1}{x_2-x_1}$, you need to find y_1 and y_2.

When $x_1=1$, $y_1=5(1)^3-3(1)^2-4(1)+7$
$$=5$$

When $x_2=-1$, $y_2=5(-1)^3-3(-1)^2-4(-1)+7$
$$=5(-1)-3(1)+4+7$$
$$=-5-3+11$$
$$=3$$

$$m=\frac{y_2-y_1}{x_2-x_1}$$
$$=\frac{3-5}{-1-1}$$
$$=\frac{-2}{-2}$$
$$=1$$

The slope of the secant is 1.

7. Since $y=x^2-3$ and $y_1=1$ and $y_2=6$, you can find x_1 and x_2.

When $y_1=1$, $1=(x_1)^2-3$
$$(x_1)^2=1+3$$
$$(x_1)^2=4$$
$$x_1=\pm 2$$

Ordinates are y-values.

When $y_2=6$, $6=(x_2)^2-3$
$$(x_2)^2=6+3$$
$$(x_2)^2=9$$
$$x_2=\pm 3$$

In order to get the equation, use the two-point formula $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$.

Equation I

$(2,1)$ and $(3,6)$

$$\frac{y-1}{x-2}=\frac{6-1}{3-2}$$
$$\frac{y-1}{x-2}=\frac{5}{1}$$
$$y-1=5x-10$$
$$5x-y-9=0$$

Equation II

$(-2,1)$ and $(-3,6)$

$$\frac{y-1}{x-(-2)}=\frac{6-1}{-3-(-2)}$$
$$\frac{y-1}{x+2}=\frac{5}{-1}$$
$$-y+1=5x+10$$
$$5x+y+9=0$$

8. You will use $m=\frac{y_2-y_1}{x_2-x_1}$ to make a table showing the slope of each secant.

a.

x	y	Slope of secant through (x,y) and $(2,4)$
1.9	$(1.9)^2$	$\frac{4-3.61}{2-1.9}=\frac{0.39}{0.1}=3.9$
1.99	$(1.99)^2$	$\frac{4-3.9601}{2-1.99}=\frac{0.0399}{0.01}=3.99$
1.999	$(1.999)^2$	$\frac{4-3.996\,001}{2-1.999}=\frac{0.003\,999}{0.001}=3.999$
⋮	⋮	

b. Since the slopes of the secants are 3.9, 3.99, 3.999,… as they approach A, the likely slope of the line at $A(2,4)$ seems to be 4.

c. The slope of the line as a multiple of a at (a,b) is $2a$.

9. As $h \to 0$, the slope of the secant decreases to $2x_1+3$.

Section 1: Activity 2

1. When $x=1$, $y=-2$.

Call $(1,-2)$ Point A, and call (x,y) Point B_n. Choose values of x greater or less than 1.

Coordinates of B_n			Slope of secant through $(1,-2)$ and $B_n(x,y)$
Point	x	$y=x^2-3$	
B_1	1.5	-0.75	$\frac{-2-(-0.75)}{1-1.5}=2.5$
B_2	1.1	-1.79	$\frac{-2-(-1.79)}{1-1.1}=2.1$
B_3	1.01	$-1.979\,9$	$\frac{-2-(-1.9799)}{1-1.01}=2.01$
B_4	1.001	$-1.997\,999$	$\frac{-2-(-1.997\,999)}{1-1.001}=2.001$

Coordinates of B_n			Slope of secant through $(1,-2)$ and $B_n(x,y)$
Point	x	$y=x^2-3$	
B_5	0.999	−2.001 999	$\frac{-2-(-2.001\,999)}{1-0.999}=1.999$
B_6	0.99	−2.019 9	$\frac{-2-(-2.0199)}{1-0.99}=1.99$
B_7	0.9	−2.19	$\frac{-2-(-2.19)}{1-0.9}=1.9$
B_8	0.5	−2.75	$\frac{-2-(-2.75)}{1-0.5}=1.5$

According to the sequence of slopes of the secants, you could conclude that the slope of the tangent at $(1,-2)$ is 2.

2. When $x=1$, $y=x^3-2x=-1$.

Call $(1,-1)$ Point A and (x,y) Point B_n. Choose values of x greater or less than 1.

Coordinates of B_n			Slope of secant through $(1,-1)$ and $B_n(x,y)$
Point	x	$y=x^3-2x$	
B_1	0.5	−0.875	$\frac{-1-(-0.875)}{1-0.5}=-0.25$

Coordinates of B_n			Slope of secant through $(1,-1)$ and $B_n(x,y)$
Point	x	$y=x^3-2x$	
B_2	0.9	−1.071	$\frac{-1-(-1.071)}{1-0.9}=0.71$
B_3	0.99	−1.009 701	$\frac{-1-(-1.009\,701)}{1-0.99}=0.9701$
B_4	0.999	−1.000 997 001	$\frac{-1-(-1.000\,997\,001)}{1-0.999}=0.997\,001$
B_5	1.001	−1.998 996 999	$\frac{-1-(0.998\,996\,999)}{1-1.001}=1.003\,001$
B_6	1.01	−0.989 699	$\frac{-1-(0.989\,699)}{1-1.01}=1.0301$
B_7	1.1	−0.869	$\frac{-1-(-0.869)}{1-1.1}=1.31$
B_8	1.5	0.375	$\frac{-1-0.375}{1-1.5}=2.75$

According to the sequence of slope of the secants, you could conclude that the slope of the tangent at $(1,-1)$ is 1.

3. $f(x)=x^2-x$ and $f(x+h)=(x+h)^2-(x+h)$

$$=x^2+2hx+h^2-x-h$$

The derivative is defined as follows:

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{since } \Delta x = h \\ &= \lim_{h\to 0}\left(x^2+2hx+h^2-x-h\right)-\left(x^2-x\right) \\ &= \lim_{h\to 0}\frac{x^2+2hx+h^2-x-h-x^2+x}{h} \\ &= \lim_{h\to 0}\frac{\cancel{h}(2x-1+h)}{\cancel{h}} \\ &= 2x-1+0 \\ &= 2x-1 \end{aligned}$$

4. The slope of the tangent at $x=1$ is $f'(1)$.

Because $f'(x)=2x-1$, $f'(1)=2(1)-1$

$$=1$$

Therefore, the slope of the tangent is 1.

The point of contact is $(1, f(1))$.

Now $f(1)=1^2-1$

$$=0$$

Therefore, the point of contact is $(1,0)$.

Since $y-y_1=m(x-x_1)$

$$\begin{aligned} y-0 &= 1(x-1) \\ y &= x-1 \end{aligned}$$

The equation of the tangent is $x-y-1=0$.

The slope of the normal is the negative reciprocal of the slope of the tangent.

$$\begin{aligned} m_{\text{normal}} &= \frac{-1}{m_{\text{tangent}}} \\ &= \frac{-1}{1} \\ &= -1 \end{aligned}$$

Using $y-y_1=m(x-x_1)$

$$\begin{aligned} y-0 &= -1(x-1) \\ y &= -x+1 \end{aligned}$$

The equation of the normal is $x+y-1=0$.

5. The slope of the tangent at $x=-1$ is $f'(-1)$.

Since $f'(x)=-3x^2$, $f'(-1)=-3(-1)^2$
$$=-3$$

Therefore, the slope of the tangent is -3.

The point of contact is $(-1, f(-1))$.

$$\begin{aligned} f(-1) &= 2-x^3 \\ &= 2-(-1)^3 \\ &= 3 \end{aligned}$$

Therefore, the point of contact is $(-1, 3)$.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-3 &= -3(x-(-1)) \\ y-3 &= -3(x+1) \\ y-3 &= -3x-3 \end{aligned}$$

The equation of the tangent is $3x+y=0$.

The slope of the normal is the negative reciprocal of the slope of the tangent.

$$\begin{aligned} m_{\text{normal}} &= \frac{-1}{m_{\text{tangent}}} \\ &= \frac{-1}{-3} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \therefore\ y-y_1 &= m(x-x_1) \\ y-3 &= \frac{1}{3}(x-(-1)) \\ 3(y-3) &= 1x+1 \\ 3y-9 &= x+1 \end{aligned}$$

The equation of the normal is $x-3y+10=0$.

Section 1: Follow-up Activities

Extra Help

1. Because $m=-\frac{2}{3}$, using $y=mx+b$, $y=-\frac{2}{3}x+b$.

   An x-intercept of 1 corresponds to the point $(1, 0)$.

   Replace x by 1 and y by 0.

$$0 = -\frac{2}{3}(1) + b$$
$$0 = -2 + 3b$$
$$3b = 2$$
$$b = \frac{2}{3}$$

Therefore, $y = -\frac{2}{3}x + \frac{2}{3}$.

Put the equation in standard form.

$$3y = 3\left(\frac{-2x}{3}\right) + 3\left(\frac{2}{3}\right)$$

Multiply each term by 3.

$$3y = -2x + 2$$

The required equation is $2x + 3y - 2 = 0$.

2. The slope m is 5.

$$y = mx + b$$
$$y = 5x + b$$

To calculate the y-intercept b, simply substitute the coordinates of the given point $(4, -3)$ into the equation.

$$-3 = 5(4) + b \quad (x = 4 \text{ and } y = -3)$$
$$b = -23$$

Substitute $b = -23$ into the slope-y-intercept equation.

$$y = 5x - 23$$

The required equation is $5x - y - 23 = 0$.

3. First, find the slope of the line.

$$(x_1, y_1) = (1, 2) \text{ and } (x_2, y_2) = (3, -2)$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - 2}{3 - 1} \\ &= \frac{-4}{2} \\ &= -2 \end{aligned}$$

The slope m is -2. Substitute into $y = mx + b$.

$$y = -2x + b$$

Use either point and replace x and y by its coordinates.

Using $(3, -2)$, $-2 = -2(3) + b$

$$-2 = -6 + b$$
$$b = 4$$

Now replace b by 4.

$$y = -2x + 4$$

In standard form the equation is $2x + y - 4 = 0$.

Enrichment

1. Because the lines are parallel, the required equation must be of the form $x + 2y + C = 0$. The only difference between the equations is in the constant terms. To evaluate C, substitute the coordinates of the given point into this form.

$$\begin{aligned} 1(1) + 2(5) + C &= 0 \\ 1 + 10 + C &= 0 \\ C &= -11 \end{aligned}$$

Replace x by 1 and y by 5.

Therefore, the required equation is $x + 2y - 11 = 0$.

2. Because the lines are parallel, the required equation must be of the form $2x - y + C = 0$. The only difference between the equations is in the constant terms. To evaluate C, substitute the coordinates of the given point into this form.

$$\begin{aligned} 2(-2) - (3) + C &= 0 \\ -4 - 3 + C &= 0 \\ C &= 7 \end{aligned}$$

Replace x by -2 and y by 3.

Therefore, the required equation is $2x - y + 7 = 0$.

3. Because the lines are perpendicular, the required equation must be of the form $2x - y + C = 0$. To evaluate C, substitute the coordinates of the given point into this form.

$$\begin{aligned} 2(1) - 1(5) + C &= 0 \\ 2 - 5 + C &= 0 \\ C &= 3 \end{aligned}$$

Replace x by 1 and y by 5.

Therefore, the required equation is $2x - y + 3 = 0$.

4. Because the lines are perpendicular, the required equation must be of the form $x + 2y + C = 0$. To evaluate C, substitute the coordinates of the given point into this form.

$$\begin{aligned} 1(-2) + 2(3) + C &= 0 \\ -2 + 6 + C &= 0 \\ C &= -4 \end{aligned}$$

Replace x by -2 and y by 3.

Therefore, the required equation is $x + 2y - 4 = 0$.

Section 2: Activity 1

1. a. $f(x) = c$

$f(x + h) = c$

The value of the function is constant; the value is independent of x.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ where } \Delta x = h$$

$$= \lim_{h \to 0} \frac{c-c}{h}$$

$$= \lim_{h \to 0} \frac{0}{h}$$

$$= \lim_{h \to 0} 0$$

$$= 0$$

b. $f(x) = x$

$f(x+h) = x + h$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1$$

$$= 1$$

c. $f(x) = x^2$

$$f(x+h) = (x+h)^2$$
$$= x^2 + 2hx + h^2$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ where } \Delta x = h$$

$$= \lim_{h \to 0} \frac{\left(x^2 + 2hx + h^2\right) - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= 2x + 0$$

$$= 2x$$

d. $f(x) = x^3$

$$f(x+h) = (x+h)^3$$
$$= (x+h)(x+h)(x+h)$$
$$= \left(x^2 + 2hx + h^2\right)(x+h)$$
$$= x^3 + 3x^2h + 3xh^2 + h^3$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{h \to 0} \frac{\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(3x^2 + 3xh + h^2\right)}{\cancel{h}}$$

$$= 3x^2 + 3x(0) + 0^2$$

$$= 3x^2$$

e. $f(x) = 2x^3 - x + 1$

$$f(x+h) = 2(x+h)^3 - (x+h) + 1$$

$$= 2\left(x^3 + 3hx^2 + 3h^2x + h^3\right) - x - h + 1$$

$$= 2x^3 + 6hx^2 + 6h^2x + 2h^3 - x - h + 1$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ where } \Delta x = h$$

$$= \lim_{h \to 0} \frac{\cancel{2x^3} + 6hx^2 + 6h^2x + 2h^3 - \cancel{x} - h + \cancel{1} - \cancel{2x^3} + \cancel{x} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(6x^2 + 6hx + 2h^2 - 1\right)}{\cancel{h}}$$

$$= \lim_{h \to 0} \left(6x^2 + 6hx + 2h^2 - 1\right)$$

$$= 6x^2 - 1 \quad \left(\text{Where } h \to 0,\ 6hx \text{ and } 2h^2 \text{ tend to } 0.\right)$$

2. In each question, the domain of f' is the set of reals. Each function is differentiable for any real value of x.

3. Because $\frac{dy}{dx}$ at $x = -2$ is a limit $\left(\frac{dy}{dx} = \lim_{h \to 0} \frac{f(-2+h) - f(2)}{h}\right)$, it only exists if the left-hand and right-hand limits are equal.

Calculate the left-hand limit at $x = -2$.

$$f(-2) = \left|(-2)^2 - 4\right|$$

$$= 0$$

$$f(-2+h) = \left|(-2+h)^2 - 4\right|$$

$$= \left|4 - 4h + h^2 - 4\right|$$

$$= \left|-4h + h^2\right|$$

$$= -4h + h^2 \quad (h < 0 \text{ and } -4h > 0\text{; that is, } h \to 0 \text{ from the left.})$$

$$\lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0^-} \frac{-4h + h^2 - 0}{h}$$

$$= \lim_{h \to 0^-} (-4 + h)$$

$$= -4$$

Calculate the right-hand limit at $x=-2$.

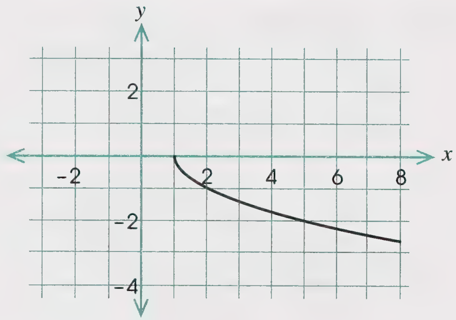
$$f(-2)=\left|(-2)^2-4\right| \\ =0$$

$$\begin{aligned} f(-2+h) &= \left|(-2+h)^2-4\right| \\ &= \left|4-4h+h^2-4\right| \\ &= \left|-4h+h^2\right| \\ &= |h| \bullet |-4+h| \\ &= h|-4+h| \quad (h>0;\text{ that is, } h \to 0 \text{ from the right.}) \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(-2+h)-f(2)}{h} &= \lim_{h\to 0^+} \frac{h|-4+h|-0}{h} \\ &= \lim_{h\to 0^+} |-4+h| \\ &= 4 \end{aligned}$$

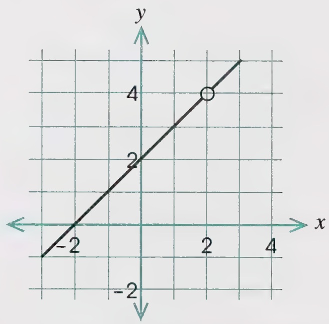
Because the left-hand limit is not equal to the right-hand limit, the derivative at $x=-2$ does not exist; that is, $f'(-2)$ is not defined. Therefore, $y=f(x)$ is not differentiable at $x=-2$.

4. a.



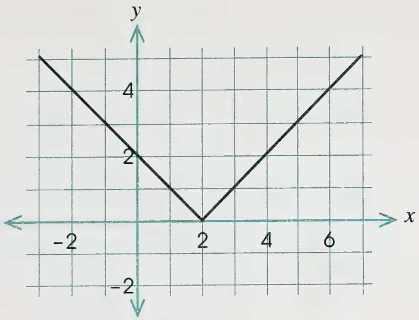
The function is not differentiable at $x=1$ since the left-hand limit for the derivative does not exist. The function is not defined for $x<1$.

b.



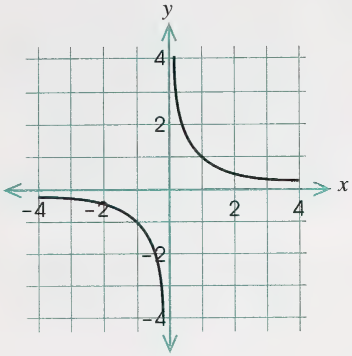
The function is not differentiable at $x=2$ since the function is not continuous at $x=2$.

c.



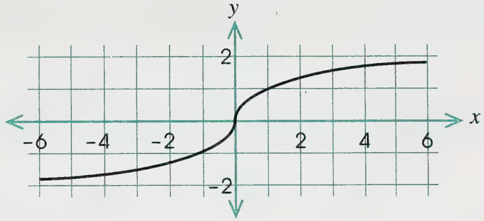
The function is not differentiable at $x = 2$ since there is a sharp transition at $x = 2$.

d.



The function is not differentiable at $x = 0$ since the function is not defined for $x = 0$.

e.



The function is not differentiable at $x = 0$ since the function has a vertical tangent at $x = 0$.

5. $$\begin{aligned} f'(4) &= 3(4) - 6 \\ &= 12 - 6 \\ &= 6 \end{aligned}$$

At $x = 4$, y is increasing by 6 when x increases by 1.

Section 2: Activity 2

1. **a.** $y = x^{12}$

$\boxed{12 - 1}$

$\frac{dy}{dx} = 12x^{11}$

b. $f(x) = x^{-5}$

$\boxed{-5 - 1}$

$f'(x) = -5x^{-6}$

c. $y = \dfrac{1}{x^{10}} = x^{-10}$

$-10-1$

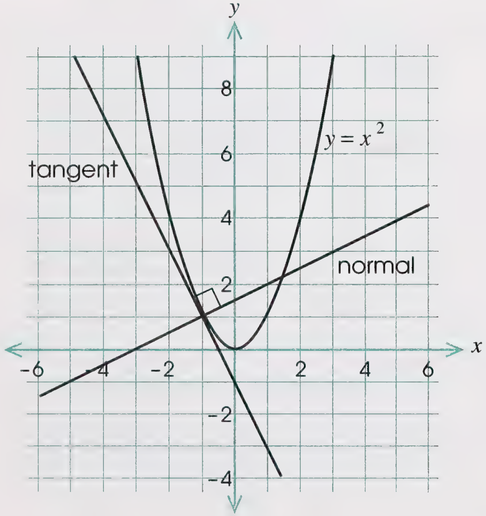
$\dfrac{dy}{dx} = -10x^{-11}$

d. $y = \sqrt[3]{x} = x^{\frac{1}{3}}$

$\dfrac{1}{3}-1$

$\dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}}$

2.



The slopes of the tangent and normal are negative reciprocals.

If the slope of the normal is $\frac{1}{2}$, the slope of the tangent is -2.

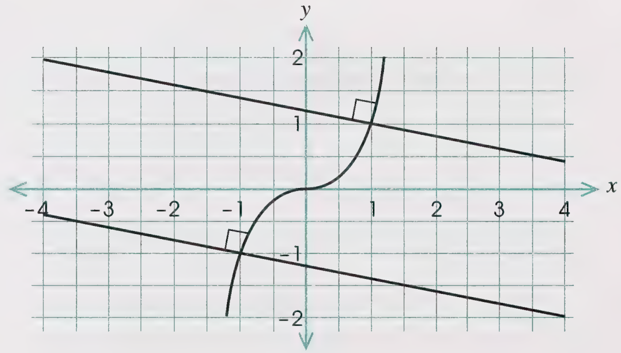
If $f(x) = x^2$, then $f'(x) = 2x$.

Use $f'(x) = 2x$ to determine the slope of the tangent.

$$\begin{aligned} 2x &= -2 \\ x &= -1 \end{aligned}$$

Since $f(-1) = (-1)^2 = 1$, the normal intersects the function at $(-1, 1)$.

3.



The derivative is $f'(x) = 5x^4$.

Since the slope of the normal is $-\frac{1}{5}$, the slope of the tangent is 5.

$$5x^4 = 5$$
$$x^4 = 1$$
$$x = \pm 1$$

There are two points on the graph at which the normal has a slope of $-\frac{1}{5}$.

Since $f(1) = 1^5 = 1$, and $f(-1) = (-1)^5 = -1$, the points are $(1, 1)$ and $(-1, -1)$.

The equation of the normal at $(1, 1)$ is

$$\begin{aligned} y - 1 &= -\frac{1}{5}(x-1) \\ 5y - 5 &= -1(x-1) \\ 5y - 5 &= -1x + 1 \\ x + 5y - 6 &= 0 \end{aligned}$$

The equation of the normal at $(-1, -1)$ is

$$\begin{aligned} y + 1 &= \frac{-1}{5}(x+1) \\ 5y + 5 &= -1(x+1) \\ 5y + 5 &= -1x - 1 \\ x + 5y + 6 &= 0 \end{aligned}$$

Section 2: Activity 3

1. a. $\frac{dy}{dx} = -7$

b. $\frac{dy}{dx} = \frac{1}{4}\left(6x^5\right)$

$= \frac{3}{2}x^5$

c. $y = -2x^{-2}$ (Write the function in the form $y = cx^n$.)

$\frac{dy}{dx} = -2(-2)x^{-2-1}$

$= 4x^{-3}$

d. $y = \frac{3}{x^3}$

$= 3x^{-3}$ (Write the function in the form $y = cx^n$.)

$\frac{dy}{dx} = 3(-3)x^{-3-1}$

$= -9x^{-4}$

e. $y = \dfrac{3}{\sqrt{x}}$

$= \dfrac{3}{x^{\frac{1}{2}}}$

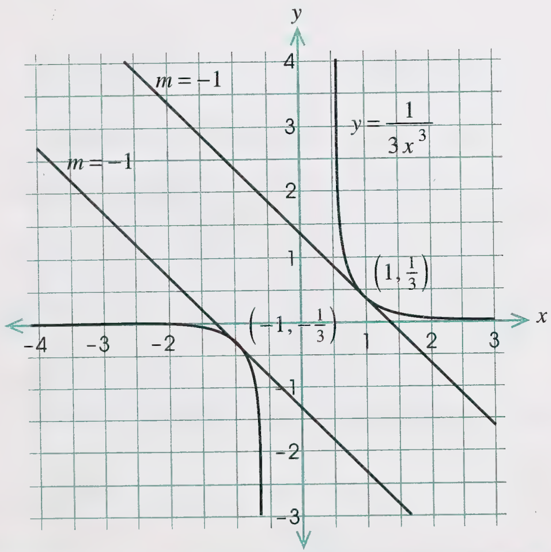
$= 3x^{-\frac{1}{2}}$ (Write the function in the form $y = cx^n$.)

$\dfrac{dy}{dx} = 3\left(-\dfrac{1}{2}\right)x^{-\frac{1}{2}-1}$

$= -\dfrac{3}{2}x^{-\frac{3}{2}}$

f. $\frac{dy}{dx} = 0$ since $y = \frac{1}{3}$ is a constant function.

2.



Express the equation with y as subject; that is, write it as $y = f(x)$.

$3x^3y = 1$

$y = \dfrac{1}{3x^3}$

$y = \dfrac{1}{3}x^{-3}$

$\dfrac{dy}{dx} = -3\left(\dfrac{1}{3}\right)x^{-3-1}$

$= -x^{-4}$

Because the slope of the curve (tangent line) is -1, $-x^{-4} = -1$.

$\dfrac{1}{x^4} = 1$

$x^4 = 1$

$x = \pm 1$

When $x = 1$, $y = \frac{1}{3}$ since $f(1) = \frac{1}{3(1)^3}$.

When $x = -1$, $y = -\frac{1}{3}$ since $f(-1) = \frac{1}{3(-1)^3}$.

The points on the curve where the slope is -1 are $\left(1, \frac{1}{3}\right)$ and $\left(-1, -\frac{1}{3}\right)$.

Section 2: Activity 4

1. a. $$\frac{dy}{dx} = (2)(4)x^3 - (3)(3)x^2 - 5$$
$$= 8x^3 - 9x^2 - 5$$

b. $$\frac{dy}{dx} = (5)x^4 - (8)(4)x^3 + (3)(3)x^2$$
$$= 5x^4 - 32x^3 + 9x^2$$

c. $$\frac{dy}{dx} = (5)(4)x^3 - \left(\frac{1}{2}\right)(3)x^2 + 0$$
$$= 20x^3 - \frac{3}{2}x^2$$

d. $$\frac{dy}{dx} = \left(\frac{1}{2}\right)(3)x^2 - (2)(2)x + 1$$
$$= \frac{3}{2}x^2 - 4x + 1$$

2. $$f(x) = x^2 - x + 1$$
$$f'(x) = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(x) + \frac{d}{dx}(1)$$
$$= 2x - 1 + 0$$
$$= 2x - 1$$

$$\therefore f'(1) = 2(1) - 1$$
$$= 1$$

Section 2: Activity 5

1. a. $$\frac{dy}{dx} = \left(x^2 - 3x + 4\right)\frac{d}{dx}\left(2x^2 - x + 1\right) + \left(2x^2 - x + 1\right)\frac{d}{dx}\left(x^2 - 3x + 4\right)$$
$$= \left(x^2 - 3x + 4\right)(4x - 1) + \left(2x^2 - x + 1\right)(2x - 3)$$
$$= 4x^3 - 12x^2 + 16x - x^2 + 3x - 4 + 4x^3 - 2x^2 + 2x - 6x^2 + 3x - 3$$
$$= 8x^3 - 21x^2 + 24x - 7$$

(You may choose to expand $\frac{dy}{dx}$.)

b. $$D_x y = \left(x^2 + 4x - 5\right)D_x\left(3x^2 + x - 1\right) + \left(3x^2 + x - 1\right)D_x\left(x^2 + 4x - 5\right)$$
$$= \left(x^2 + 4x - 5\right)(6x + 1) + \left(3x^2 + x - 1\right)(2x + 4)$$
$$= 6x^3 + 24x^2 - 30x + x^2 + 4x - 5 + 6x^3 + 2x^2 - 2x + 12x^2 + 4x - 4$$
$$= 12x^3 + 39x^2 - 24x - 9$$

c. $D_x y = x^{-2} D_x (x+3) + (x+3) D_x \left(x^{-2}\right)$

$$= x^{-2}(1) + (x+3)(-2)x^{-3}$$

$$= x^{-2} - 2(x+3)x^{-3}$$

d. $D_x y = x^{-5} D_x (x-7) + (x-7) D_x \left(x^{-5}\right)$

$$= x^{-5}(1) + (x-7)(-5)x^{-6}$$

$$= x^{-5} - 5(x-7)x^{-6}$$

$$= -4x^{-5} + 35x^{-6}$$

2. $y = (x+1)(x-5)$

$$\frac{dy}{dx} = (x+1)\frac{d}{dx}(x-5) + (x-5)\frac{d}{dx}(x+1)$$

$$= (x+1)(1) + (x-5)(1)$$

$$= 2x - 4$$

Now, multiply first and compare the derivatives.

$$y = x^2 - 4x - 5$$

$$\frac{dy}{dx} = 2x - 4$$

The derivatives are the same.

Section 2: Activity 6

1. a. Let $u = 5x - 3$.

$$\frac{du}{dx} = 5$$

$$y = u^6$$

$$\frac{dy}{du} = 6u^5$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$= 6u^5 \bullet 5$$

$$= 30u^5$$

$$= 30(5x-3)^5$$

b. Let $u = 6x + 7$.

$$\frac{du}{dx} = 6$$

$$y = u^7$$

$$\frac{dy}{dx} = 7u^6$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$= 7u^6(6)$$

$$= 42(6x+7)^6$$

c. $y = (x-5)^{-1}$

Let $u = x - 5$.

$$\frac{du}{dx} = 1$$

$$y = u^{-1}$$

$$\frac{dy}{du} = -1u^{-2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$= -1u^{-2}(1)$$

$$= -\frac{1}{u^2}$$

$$= -\frac{1}{(x-5)^2}$$

d. $y = 4\left(x^2 + 2\right)^{-1}$

Let $u = x^2 + 2$.

$\frac{du}{dx} = 2x$

$y = 4u^{-1}$

$\frac{dy}{du} = -4u^{-2}$

$$\begin{aligned}\frac{dy}{dx} &= -4u^{-2} \bullet 2x \\ &= -8xu^{-2} \\ &= -8x\left(x^2+2\right)^{-2}\end{aligned}$$

e. $y = (3x-1)^{\frac{1}{2}}$

Let $u = 3x - 1$.

$\frac{du}{dx} = 3$

$y = u^{\frac{1}{2}}$

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \bullet \frac{du}{dx} \\ &= \frac{1}{2}u^{-\frac{1}{2}} \bullet 3 \\ &= \frac{3}{2}u^{-\frac{1}{2}} \\ &= \frac{3}{2u^{\frac{1}{2}}} \\ &= \frac{3}{2\sqrt{3x-1}}\end{aligned}$$

f. $y = (4x+3)^{\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(4x+3)^{-\frac{1}{2}}(4) \\ &= 2(4x+3)^{-\frac{1}{2}}\end{aligned}$$

g. $y = 3(x-2)^{-\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= (3)\left(-\frac{1}{2}\right)(x-2)^{-\frac{3}{2}}(1) \\ &= -\frac{3}{2}(x-2)^{-\frac{3}{2}}\end{aligned}$$

h. $y = 5(2x+1)^{-\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= (5)\left(-\frac{1}{2}\right)(2x+1)^{-\frac{3}{2}}(2) \\ &= -5(2x+1)^{-\frac{3}{2}}\end{aligned}$$

i. Let $u = 20 - 7x^2$.

$\frac{du}{dx} = -14x$

$y = u^5$

$\frac{dy}{du} = 5u^4$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \bullet \frac{du}{dx} \\ &= 5u^4(-14x) \\ &= -70xu^4 \\ &= -70x\left(20-7x^2\right)^4\end{aligned}$$

j. $y = 3x^{-\frac{5}{3}}$

$$D_x y = (3)\left(-\frac{5}{3}\right)x^{-\frac{5}{3}-1}$$

$$= -5x^{-\frac{8}{3}}$$

2. $\frac{dr}{dt} = 4$

$C = 2\pi r$

$\frac{dC}{dr} = 2\pi$

$$\frac{dC}{dt} = \frac{dC}{dr} \bullet \frac{dr}{dt}$$

$$= 2\pi(4)$$

$$= 8\pi \text{ cm/s}$$

The circumference of the balloon is changing at 8π cm/s.

3. **a.** Let $u = 3x - 1$, $y = u^8$

$\frac{du}{dx} = 3$

$$y' = nu^{n-1}\frac{du}{dx}$$

$$= 8(3x-1)^7(3)$$

$$= 24(3x-1)^7$$

b. $y = (2x-1)^{-10}$

Let $u = 2x - 1$; $\frac{du}{dx} = 2$

$y = u^{-10}$

$$y' = nu^{n-1}\frac{du}{dx}$$

$$= -10(2x-1)^{-10-1}(2)$$

$$= -20(2x-1)^{-11}$$

$$= \frac{-20}{(2x-1)^{11}}$$

c. $y = (4-x)^{\frac{1}{2}}$

$$y' = \frac{1}{2}(4-x)^{\frac{1}{2}-1}(-1)$$

$$= -\frac{1}{2}(4-x)^{-\frac{1}{2}}$$

$$= -\frac{1}{2(4-x)^{\frac{1}{2}}}$$

$$= -\frac{1}{2\sqrt{4-x}}$$

4. **a.** $D_x y = (x-3)^{-2}D_x(x+1) + (x+1)D_x(x-3)^{-2}$

$$= (x-3)^{-2}(1) + (x+1)(-2)(x-3)^{-3}(1)$$

$$= (x-3)^{-2} - 2(x+1)(x-3)^{-3}$$

$$= (x-3)^{-3}\left[(x-3)^1 - 2(x+1)\right]$$

$$= (x-3)^{-3}[x-3-2x-2]$$

$$= (x-3)^{-3}(-x-5)$$

b. $$\begin{aligned}D_x y &= (x+4)^{-3} D_x (x-2) + (x-2) D_x (x+4)^{-3}\\ &= (x+4)^{-3}(1) + (x-2)(-3)(x+4)^{-4}(1)\\ &= (x+4)^{-3} - 3(x-2)(x+4)^{-4}\\ &= (x+4)^{-4}\left[(x+4)^{1} - 3(x-2)\right]\\ &= (x+4)^{-4}[x+4-3x+6]\\ &= (x+4)^{-4}(-2x+10)\\ &= -2(x+4)^{-4}(x-5)\end{aligned}$$

c. $$\begin{aligned}D_x y &= (x-5) D_x \sqrt{x+2} + \left(\sqrt{x+2}\right) D_x (x-5)\\ &= (x-5)\left(\frac{1}{2}\right)(x+2)^{-\frac{1}{2}}(1) + (x+2)^{\frac{1}{2}}(1)\\ &= \frac{1}{2}(x-5)(x+2)^{-\frac{1}{2}} + (x+2)^{\frac{1}{2}}\\ &= \frac{1}{2}(x+2)^{-\frac{1}{2}}[(x-5)+2(x+2)]\\ &= \frac{1}{2}(x+2)^{-\frac{1}{2}}(3x-1)\end{aligned}$$

d. $$\begin{aligned}D_x y &= (x-3)^{\frac{1}{2}} D_x (x+1) + (x+1) D_x (x-3)^{\frac{1}{2}}\\ &= (x-3)^{\frac{1}{2}}(1) + (x+1)\left(\frac{1}{2}\right)(x-3)^{-\frac{1}{2}}(1)\\ &= (x-3)^{\frac{1}{2}} + \frac{1}{2}(x+1)(x-3)^{-\frac{1}{2}}\\ &= \frac{1}{2}(x-3)^{-\frac{1}{2}}[2(x-3)+(x+1)]\\ &= \frac{1}{2}(x-3)^{-\frac{1}{2}}(3x-5)\end{aligned}$$

e. $$\begin{aligned}D_x y &= (x-2)^{\frac{1}{2}} D_x (x+1)^{\frac{1}{2}} + (x+1)^{\frac{1}{2}} D_x (x-2)^{\frac{1}{2}}\\ &= (x-2)^{\frac{1}{2}}\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}}(1) + (x+1)^{\frac{1}{2}}\left(\frac{1}{2}\right)(x-2)^{-\frac{1}{2}}(1)\\ &= \frac{1}{2}(x+1)^{-\frac{1}{2}}(x-2)^{-\frac{1}{2}}(x-2+x+1)\\ &= \frac{2x-1}{2(x+1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}}\end{aligned}$$

f. $$\begin{aligned}D_x y &= (x-3)^{\frac{1}{2}} D_x (x-5)^{\frac{1}{2}} + (x-5)^{\frac{1}{2}} D_x (x-3)^{\frac{1}{2}}\\ &= (x-3)^{\frac{1}{2}}\left(\frac{1}{2}\right)(x-5)^{-\frac{1}{2}}(1) + (x-5)^{\frac{1}{2}}\left(\frac{1}{2}\right)(x-3)^{-\frac{1}{2}}(1)\\ &= \frac{1}{2}(x-5)^{-\frac{1}{2}}(x-3)^{-\frac{1}{2}}(x-3+x-5)\\ &= \frac{2x-8}{2(x-5)^{\frac{1}{2}}(x-3)^{\frac{1}{2}}}\end{aligned}$$

Section 2: Activity 7

1. a.

$$\frac{dy}{dx}=\frac{(5x+7)\frac{d}{dx}(3x-2)-(3x-2)\frac{d}{dx}(5x+7)}{(5x+7)^2}$$

$$=\frac{(5x+7)(3)-(3x-2)(5)}{(5x+7)^2}$$

$$=\frac{15x+21-15x+10}{(5x+7)^2}$$

$$=\frac{31}{(5x+7)^2}$$

b.

$$\frac{dy}{dx}=\frac{\left(5+x+2x^2\right)\frac{d}{dx}\left(2-3x+x^2\right)-\left(2-3x+x^2\right)\frac{d}{dx}\left(5+x+2x^2\right)}{\left(5+x+2x^2\right)^2}$$

$$=\frac{\left(5+x+2x^2\right)(-3+2x)-\left(2-3x+x^2\right)(1+4x)}{\left(5+x+2x^2\right)^2}$$

$$=\frac{\left(-15-3x-6x^2+10x+2x^2+4x^3\right)-\left(2-3x+x^2+8x-12x^2+4x^3\right)}{\left(5+x+2x^2\right)^2}$$

$$=\frac{7x^2+2x-17}{\left(5+x+2x^2\right)^2}$$

c. $$\frac{dy}{dx}=\frac{\left(3x-x^2\right)\frac{d}{dx}(5x-1)-(5x-1)\frac{d}{dx}\left(3x-x^2\right)}{\left(3x-x^2\right)^2}$$

$$=\frac{\left(3x-x^2\right)(5)-(5x-1)(3-2x)}{\left(3x-x^2\right)^2}$$

$$=\frac{15x-5x^2-15x+3+10x^2-2x}{\left(3x-x^2\right)^2}$$

$$=\frac{5x^2-2x+3}{\left(3x-x^2\right)^2}$$

d. $$\frac{dy}{dx}=\frac{\left(\sqrt{2x^2+1}\right)\frac{d}{dx}(x)-(x)\frac{d}{dx}\sqrt{2x^2+1}}{\left(2x^2+1\right)}$$

$$=\frac{\sqrt{2x^2+1}-(x)\left(\frac{1}{2}\right)\left(2x^2+1\right)^{-\frac{1}{2}}(4x)}{\left(2x^2+1\right)}$$

$$=\frac{\left(2x^2+1\right)^{\frac{1}{2}}-2x^2\left(2x^2+1\right)^{-\frac{1}{2}}}{\left(2x^2+1\right)}$$

$$=\frac{\left(2x^2+1\right)^{-\frac{1}{2}}\left(\cancel{2x^2}+1-\cancel{2x^2}\right)}{\left(2x^2+1\right)}$$

$$=\frac{\left(2x^2+1\right)^{-\frac{1}{2}}}{\left(2x^2+1\right)^1}$$

$$=\frac{1}{\left(2x^2+1\right)^{\frac{3}{2}}}$$

e. $$\frac{dy}{dx}=\frac{\left(x^2-3\right)^{\frac{1}{2}}\frac{d}{dx}(2x-3)-(2x-3)\frac{d}{dx}\left(x^2-3\right)^{\frac{1}{2}}}{x^2-3}$$

$$=\frac{\left(x^2-3\right)^{\frac{1}{2}}(2)-(2x-3)\left(\tfrac{1}{2}\right)\left(x^2-3\right)^{-\frac{1}{2}}(2x)}{x^2-3}$$

$$\frac{\left(x^2-3\right)^{\frac{1}{2}}(2)-x(2x-3)\left(x^2-3\right)^{-\frac{1}{2}}}{x^2-3}$$

$$=\frac{\left(x^2-3\right)^{-\frac{1}{2}}\left[\left(x^2-3\right)(2)-x(2x-3)\right]}{x^2-3}$$

$$\frac{\left(x^2-3\right)^{-\frac{1}{2}}\left(2x^2-6-2x^2+3x\right)}{x^2-3}$$

$$=\frac{\left(x^2-3\right)^{-\frac{1}{2}}(3x-6)}{\left(x^2-3\right)}$$

$$=\frac{3x-6}{\left(x^2-3\right)^{\frac{3}{2}}}$$

f. $$\frac{dy}{dx}=\frac{\left(3-x^2\right)^{\frac{1}{2}}\frac{d}{dx}(-5x)-(-5x)\frac{d}{dx}\left(3-x^2\right)^{\frac{1}{2}}}{\left(3-x^2\right)}$$

$$=\frac{\left(3-x^2\right)^{\frac{1}{2}}(-5)-(-5x)\left(\tfrac{1}{2}\right)\left(3-x^2\right)^{-\frac{1}{2}}(-2x)}{\left(3-x^2\right)}$$

$$=\frac{-5\left(3-x^2\right)^{\frac{1}{2}}-5x^2\left(3-x^2\right)^{-\frac{1}{2}}}{\left(3-x^2\right)}$$

$$=\frac{\left(3-x^2\right)^{-\frac{1}{2}}\left[-5\left(3-x^2\right)-5x^2\right]}{\left(3-x^2\right)}$$

$$=\frac{\left(3-x^2\right)^{-\frac{1}{2}}\left(-15+5x^2-5x^2\right)}{\left(3-x^2\right)}$$

$$=\frac{-15}{\left(3-x^2\right)^{\frac{3}{2}}}$$

2. $y = \frac{x^5}{x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2 \bullet \frac{d}{dx}\left(x^5\right) - x^5 \bullet \frac{d}{dx}\left(x^2\right)}{\left(x^2\right)^2}\\ &= \frac{x^2\left(5x^4\right) - x^5(2x)}{x^4}\\ &= \frac{3x^6}{x^4}\\ &= 3x^2\end{aligned}$$

If you divide first, $y = x^3$ and $\frac{dy}{dx} = 3x^2$. The answers are the same.

c.

$$\begin{aligned}\frac{x^{\frac{1}{3}}}{a} - \frac{y^{\frac{1}{3}}}{a} &= 1\\ x^{\frac{1}{3}} - y^{\frac{1}{3}} &= a\\ \frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{3}y^{-\frac{2}{3}}y' &= 0\\ x^{-\frac{2}{3}} - y^{-\frac{2}{3}}y' &= 0\\ y' &= \frac{-x^{-\frac{2}{3}}}{-y^{-\frac{2}{3}}}\\ &= \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\end{aligned}$$

Section 2: Activity 8

1. a.

$$\begin{aligned}3xy + 2x^2 + y^2 &= 2\\ 3xy' + 3y + 4x + 2yy' &= 0\\ y' &= \frac{-3y - 4x}{3x + 2y}\end{aligned}$$

b.

$$\begin{aligned}y^3 - 3xy^2 - 2x^2y + x &= 2\\ 3y^2y' - 3x(2yy') - 3y^2 - 2x^2y' - 4xy + 1 &= 0\\ y' &= \frac{3y^2 + 4xy - 1}{3y^2 - 6xy - 2x^2}\end{aligned}$$

d.

$$\begin{aligned}xy^2 &= 2x - \frac{1}{x}\\ x(2yy') + y^2 &= 2 + \frac{1}{x^2}\\ y' &= \frac{\frac{2x^2+1}{x^2} - y^2}{2xy}\\ &= \frac{2x^2 + 1 - x^2y^2}{2x^3y}\end{aligned}$$

2. $$x^2 + 4xy - 2y = 3$$

$$2x + 4xy' + 4y - 2y' = 0$$

$$y' = \frac{-2x - 4y}{4x - 2}$$

$$= \frac{-x - 2y}{2x - 1}$$

Solve for y' at $P(1,1)$.

$$m = \frac{-3}{1}$$

Substitute $m = \frac{-3}{1}$ and $P(1,1)$ in the slope-point formula.

$$y - 1 = \frac{-3}{1}(x - 1)$$

$$3x + y - 4 = 0$$

Therefore, the tangent to the graph is $3x + y - 4 = 0$.

3. $$x^2 + y^2 = 169$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

At $P(5,12)$ the slope $\frac{dy}{dx} = -\frac{5}{12}$.

Thus, substitute $P(5,12)$ and $m = \frac{-5}{12}$ in the slope-point formula.

$$y - 12 = \frac{-5}{12}(x - 5)$$

$$5x + 12y = 169$$

The tangent equation at P is $5x + 12y = 169$. (1)

At $Q(5,-12)$ the slope $\frac{dy}{dx} = -\frac{5}{-12} = \frac{5}{12}$.

Therefore, substitute $Q(5,-12)$ and $m = \frac{5}{12}$ in the point-slope formula.

$$y - (-12) = \frac{5}{12}(x - 5)$$

$$5x - 12y = 169$$

The tangent equation at Q is $5x - 12y = 169$. (2)

In order to find Point T where the tangents meet, solve equations (1) and (2).

$$\begin{aligned} & 5x + 12y = 169 \quad (1) \\ & 5x - 12y = 169 \quad (2) \\ (1) + (2): \quad & 10x = 338 \\ & x = 33.8 \end{aligned}$$

When $x = 33.8,\ y = 0$.

Therefore, the Point T at where the tangents meet is $(33.8, 0)$.

Section 2: Activity 9

1. $f'(x) = 5(3x^2) - 3(1)$
$\quad = 15x^2 - 3$

$f''(x) = 15(2x) - 0$
$\quad = 30x$

$f'''(x) = 30(1)$
$f'''(x) = 30$
$f'''(2) = 30$

2. Each time the derivative is taken, the degree of the polynomial is reduced by 1. An n^{th}-degree polynomial is reduced to degree zero when the derivative is taken n times. A polynomial of degree zero is a constant. Taking the derivative one more time yields zero.

$f^{(n+1)}(x) = 0$

3. $y = (x-1)\sqrt{2x+1}$
$\quad = (x-1)(2x+1)^{\frac{1}{2}}$

Apply the product rule.

$$\begin{aligned}
y' &= (x-1)\frac{d}{dx}(2x+1)^{\frac{1}{2}} + (2x+1)^{\frac{1}{2}}\frac{d}{dx}(x-1) \\
&= (x-1)\left(\frac{1}{2}\right)(2x+1)^{\frac{1}{2}-1}(2) + (2x+1)^{\frac{1}{2}} \\
&= (x-1)(2x+1)^{-\frac{1}{2}} + (2x+1)^{\frac{1}{2}} \\
&= (2x+1)^{-\frac{1}{2}}\left[(x-1) + (2x+1)^{1}\right] \\
&= 3x(2x+1)^{-\frac{1}{2}}
\end{aligned}$$

Apply the product rule again.

$$\begin{aligned}
y'' &= 3x\frac{d}{dx}(2x+1)^{-\frac{1}{2}} + (2x+1)^{-\frac{1}{2}}\frac{d}{dx}(3x) \\
&= 3x\left(\frac{-1}{2}\right)(2x+1)^{-\frac{1}{2}-1}(2) + (2x+1)^{-\frac{1}{2}}(3) \\
&= -3x(2x+1)^{-\frac{3}{2}} + 3(2x+1)^{-\frac{1}{2}} \\
&= 3(2x+1)^{-\frac{3}{2}}\left[-x + (2x+1)\right] \\
&= 3(2x+1)^{-\frac{3}{2}}(-x + 2x + 1) \\
&= 3(2x+1)^{-\frac{3}{2}}(x+1)
\end{aligned}$$

4. $y = x^5$

$y' = 5x^4$

$y'' = 5(4)x^3$

$y''' = 5(4)(3)x^2$

$y^{(4)} = 5(4)(3)(2x)$

$y^{(5)} = 5(4)(3)(2)(1) = 5!$

5. $y^2 + y = x$

$2yy' + y' = 1$

$(2y+1)y' = 1$

$$y' = \frac{1}{2y+1}$$

Now $y' = (2y+1)^{-1}$.

$\therefore\ y'' = -1(2y+1)^{-2}(2y')$

Substitute for y'.

$$y'' = \frac{-1}{(2y+1)^2} \bullet \frac{2}{(2y+1)}$$

$$= \frac{-2}{(2y+1)^3}$$

6. $y^2 = x^3$

$2yy' = 3x^2$

$$y' = \frac{3x^2}{2y}$$

$$y' = \frac{3x^2}{2}y^{-1}$$

$$y'' = \frac{3x^2}{2}\left(-1y^{-2}y'\right) + y^{-1}\left(\frac{3}{2}\right)(2x)$$

$$= -\frac{3}{2}x^2y^{-2}y' + 3xy^{-1}$$

$$= \frac{-3x^2}{2y^2}y' + \frac{3x}{y}$$

Replace y' by $\frac{3x^2}{2y}$.

$$y'' = \frac{-3x^2\left(3x^2\right)}{2y^2(2y)} + \frac{3x}{y}$$

$$= \frac{-9x^4 + 12xy^2}{4y^3}$$

Section 2: Follow-up Activities

Extra Help

1. a. $x^{-3}(2x-1)+x^{-4}(2x-1)^2 = x^{-4}(2x-1)[x+(2x-1)]$

$= x^{-4}(2x-1)[3x-1]$

b. $x^{\frac{1}{2}}(x+1)^{-2}+x^{\frac{3}{2}}(x+1)^{-1} = x^{\frac{1}{2}}(x+1)^{-2}[1+x(x+1)]$

$= x^{\frac{1}{2}}(x+1)^{-2}\left[1+x^2+x\right]$

$= x^{\frac{1}{2}}(x+1)^{-2}\left(x^2+x+1\right)$

c. $\left(\sqrt{x+1}\right)(2x-3)^{-1}+\frac{1}{\sqrt{x+1}}(2x-3)^{-2}$

$=(x+1)^{\frac{1}{2}}(2x-3)^{-1}+(x+1)^{-\frac{1}{2}}(2x-3)^{-2}$

$=(x+1)^{-\frac{1}{2}}(2x-3)^{-2}\left[(x+1)(2x-3)+1\right]$

$=(x+1)^{-\frac{1}{2}}(2x-3)^{-2}\left[\left(2x^2-x-3\right)+1\right]$

$=(x+1)^{-\frac{1}{2}}(2x-3)^{-2}\left[2x^2-x-2\right]$

2. $y=x^2(x-3)^{-1}(x+1)^3$

Treat $x^2(x-3)^{-1}$ as a single factor.

$$\frac{dy}{dx} = x^2(x-3)^{-1}\frac{d}{dx}(x+1)^3+(x+1)^3\frac{d}{dx}\left[x^2(x-3)^{-1}\right]$$

$$= x^2(x-3)^{-1}(3)(x+1)^2$$

$$+(x+1)^3\left[x^2(-1)(x-3)^{-2}+2x(x-3)^{-1}\right]$$

$$= 3x^2(x-3)^{-1}(x+1)^2-x^2(x-3)^{-2}(x+1)^3$$

$$+2x(x-3)^{-1}(x+1)^3$$

$$= x(x-3)^{-2}(x+1)^2\left[3x(x-3)-x(x+1)+2(x-3)(x+1)\right]$$

$$= x(x-3)^{-2}(x+1)^2\left[3x^2-9x-x^2-x+2x^2-4x-6\right]$$

$$= x(x-3)^{-2}(x+1)^2\left[4x^2-14x-6\right]$$

$$= x(x-3)^{-2}(x+1)^2(2)\left(2x^2-7x-3\right)$$

$$= 2x(x-3)^{-2}(x+1)^2\left(2x^2-7x-3\right)$$

Enrichment

1. $\frac{d^3y}{dx^3}=u\bullet\frac{d^3v}{dx^3}+3\frac{du}{dx}\bullet\frac{d^2v}{dx^2}+3\frac{d^2u}{dx^2}\bullet\frac{dv}{dx}+v\frac{d^3u}{dx^3}$

To differentiate, apply the product rule to each term.

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(u \bullet \frac{d^3v}{dx^3}\right) + \frac{d}{dx}\left(3 \bullet \frac{du}{dx} \bullet \frac{d^2v}{dx^2}\right) + \frac{d}{dx}\left(3 \bullet \frac{d^2u}{dx^2} \bullet \frac{dv}{dx}\right)$$

$$+ \frac{d}{dx}\left(v\frac{d^3u}{dx^3}\right)$$

$$= u\frac{d^4v}{dx^4} + \frac{du}{dx} \bullet \frac{d^3v}{dx^3} + 3\left[\frac{d^2u}{dx^2} \bullet \frac{d^2v}{dx^2} + \frac{du}{dx} \bullet \frac{d^3v}{dx^3}\right]$$

$$+3\left[\frac{d^2u}{dx^2} \bullet \frac{d^2v}{dx^2} + \frac{d^3u}{dx^3} \bullet \frac{dv}{dx}\right] + \frac{dv}{dx} \bullet \frac{d^3u}{dx^3} + v\frac{d^4u}{dx^4}$$

$$= u\frac{d^4v}{dx^4} + \frac{du}{dx} \bullet \frac{d^3v}{dx^3} + 3 \bullet \frac{d^2u}{dx^2} \bullet \frac{d^2v}{dx^2} + 3 \bullet \frac{du}{dx} \bullet \frac{d^3v}{dx^3}$$

$$+3 \bullet \frac{d^2u}{dx^2} \bullet \frac{d^2v}{dx^2} + 3 \bullet \frac{d^3u}{dx^3} \bullet \frac{dv}{dx} + \frac{dv}{dx} \bullet \frac{d^3u}{dx^3} + v\frac{d^4u}{dx^4}$$

$$= u\frac{d^4v}{dx^4} + 4 \bullet \frac{du}{dx} \bullet \frac{d^3v}{dx^3} + 6 \bullet \frac{d^2u}{dx^2} \bullet \frac{d^2v}{dx^2} + 4 \bullet \frac{d^3u}{dx^3} \bullet \frac{dv}{dx} + v\frac{d^4u}{dx^4}$$

2. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$$\frac{d^4(uv)}{dx^4} = u\frac{d^4v}{dx^4} + 4 \bullet \frac{du}{dx} \bullet \frac{d^3v}{dx^3} + 6 \bullet \frac{d^2u}{dx^2} \bullet \frac{d^2v}{dx^2} + 4 \bullet \frac{d^3u}{dx^3} \bullet \frac{dv}{dx}$$

$$+ v\frac{d^4u}{dx^4}$$

The coefficients are identical and the orders of the derivatives correspond to the exponents of *a* and *b* in the binomial expansion.

3. $(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

The coefficients are 1, 5, 10, 10, 5, and 1.

4. $y = x^3(1+2x)^{\frac{1}{2}}$

Let $u = x^3$.

$$\therefore \frac{du}{dx} = 3x^2$$

$$\therefore \frac{d^2u}{dx^2} = 6x$$

Let $v = (1+2x)^{\frac{1}{2}}$.

$$\therefore \frac{dv}{dx} = \frac{1}{2}(1+2x)^{-\frac{1}{2}}(2)$$

$$= (1+2x)^{-\frac{1}{2}}$$

$$\therefore \frac{d^2v}{dx^2} = -\frac{1}{2}(1+2x)^{-\frac{3}{2}}(2)$$

$$= -(1+2x)^{-\frac{3}{2}}$$

Substitute into

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2 \bullet \frac{du}{dx} \bullet \frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

$$\therefore \frac{d^2y}{dx^2} = x^3\left[-(1+2x)^{-\frac{3}{2}}\right] + 2\left(3x^2\right)(1+2x)^{-\frac{1}{2}}$$

$$+(1+2x)^{\frac{1}{2}} \bullet 6x$$

$$= -x^3(1+2x)^{-\frac{3}{2}} + 6x^2(1+2x)^{-\frac{1}{2}} + 6x(1+2x)^{\frac{1}{2}}$$

$$= x(1+2x)^{-\frac{3}{2}}\left[-x^2 + 6x(1+2x) + 6(1+2x)^2\right]$$

$$= x(1+2x)^{-\frac{3}{2}}\left[-x^2 + 6x + 12x^2 + 6\left(1+4x+4x^2\right)\right]$$

$$= x(1+2x)^{-\frac{3}{2}}\left[-x^2 + 6x + 12x^2 + 6 + 24x + 24x^2\right]$$

$$= x(1+2x)^{-\frac{3}{2}}\left[35x^2 + 30x + 6\right]$$

5. $y = x(x+1)^{-1}$

Let $u = x$.

$$\therefore \frac{du}{dx} = 1$$

$$\therefore \frac{d^2u}{dx^2} = 0$$

$$\therefore \frac{d^3u}{dx^2} = 0$$

Let $v = (x+1)^{-1}$.

$$\therefore \frac{dv}{dx} = -1(x+1)^{-2}$$

$$\therefore \frac{d^2v}{dx^2} = 2(x+1)^{-3}$$

$$\therefore \frac{d^3v}{dx^2} = -6(x+1)^{-4}$$

Substitute into

$$\frac{d^3y}{dx^2} = u\frac{d^3v}{dx^3} + 3\bullet\frac{du}{dx}\bullet\frac{d^2v}{dx^2} + 3\bullet\frac{d^2u}{dx^2}\bullet\frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

$$\therefore \frac{d^3y}{dx^2} = x(-6)(x+1)^{-4} + 3(1)(2)(x+1)^{-3}$$

$$+3(0)(-1)(x+1)^{-2} + (x+1)^{-1}(0)$$

$$= -6x(x+1)^{-4} + 6(x+1)^{-3}$$

$$= 6(x+1)^{-4}\left[-x + (x+1)\right]$$

$$= 6(x+1)^{-4}$$

$$= \frac{6}{(x+1)^4}$$

NOTES

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